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The Tree of Mathematics

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RELAXATION METHODS

William Squire

INTRODUCTION

The problem of solving simultaneous linear equations arises in many branches of science and technology. Theoretically the problem is simple and an exact solution can be obtained in a straightforward manner. In practice, however, the numerical work is extremely difficult when more than four or five equations are involved. Considerable ingenuity has been employed in developing techniques for handling such problems and a number of special computers are available.

An extremely important method of handling very large numbers of equations is the so-called "relaxation" technique. The basic idea of the method is quite old, having been traced back to the great nineteenth century mathematician Gauss. However, the current popularity and many of the applications are due to the work of Southwell and his school. The name "relaxation" comes from the terminology of the structural problems to which Southwell first applied the method.

The relaxation method gives an approximate solution of the system of equations. The approximation can be made as close as desired. Naturally the greater the accuracy desired the more work is involved. In all physical problems the data is only known to a limited degree of accuracy and the solution cannot be more accurate than the data. Any work which gives an answer which appears more precise involves wasted effort.

A second feature of the method is that it gives full scope to the judgment of the computer. For example, the method starts by guessing a solution to the problem. No matter how poor the initial guess is, the correct answer will be obtained ultimately. But the better the initial guess, the less work will be involved. Also as the relaxation process is essentially a modification of the original guess toward the correct solution, the ability of the computer to understand the problems can be an important factor in speeding up the process.

Another important advantage of the process is that it is self-checking. It is possible to use the results of work which is marred by arithmetical errors and start over in such a way that these errors will be automatically eliminated.

Because of this self-checking feature and its utilization of the computer's judgment, relaxation technique enables the computer using pencil and paper to compete with the modern high-speed machine in many cases. The machine performs arithmetical operations with incredible speed and it can exercise some judgment in deciding between alternative procedures on the basis of definite instructions, but this slows it up considerably. It, therefore, attacks problems by brute force and in some cases may follow a long spiraling path toward the solution. In these cases the much slower human computer who can select a direct path remains in the running.

DESCRIPTION OF METHOD

The simplest way to explain the method is by working a simple problem. For such a simple case the use of the relaxation technique will require more work than an exact method. The labor-saving advantage only shows up for large systems.

Consider the set of equations

$$X + 4Y = 6$$
$$2X - Y = 3$$

It is easily found by elimination that $X = 2$ and $Y = 1$.

The fundamental idea of the relaxation method is to rewrite the equations as

$$X + 4Y - 6 = R_1$$
$$2X - Y - 3 = R_2$$

and to focus attention on the quantities R_1 and R_2 which are called residuals. The first step is to construct a table showing how R_1 and R_2 change when X and Y are changed. This table (Table I) is called an operation table.

Table I

	ΔR_1	ΔR_2
ΔX	1	2
ΔY	4	-1

An arbitrary guess is made for the unknowns and the residuals are calculated. When the correct values are found the residuals will vanish. In general, the initial guess will not be correct, so it is necessary to reduce the residuals by changing the values of the unknown in accordance with the operations table. This process is called "relaxing" the residuals. When the residuals become negligibly small the changes made in the unknowns are added up and the corrected values obtained. The work is then checked by recalculating the residuals. If errors have been made the new values will not be negligible. However, unless some systematic error has been made the residuals will have been reduced. By using the revised values as a starting point, the correct solution can rapidly be obtained by further relaxation without locating the numerical errors.

The solution of the sample equations is shown in Table II. As our initial guess it is assumed that X and Y are zero, giving $R_1 = -6$ and

$R_2 = -3$.

Table II

$X_i = 0$	R_1	R_2
$Y_i = 0$	-6.00	-3.00
Operation		
$\Delta Y = 1.50$	.00	-4.50
$\Delta X = 2.25$	2.25	.00
$\Delta Y = -.50$	.25	.50
$\Delta X = -.25$	.00	.00

The first step is to relax the largest residual R_1 by changing Y by 1.50. Then R_2 is relaxed by changing X by 2.25. The nearest simple value $-.50$ is then used on R_1. Judgment enables us to see that if we do not quite cancel out R_1, when we relax R_2, what we have left will be reduced. This is called under-relaxation, while the reverse process of changing an unknown by a greater quantity than needed to cancel the residual is called over-relaxation. This is a device that speeds up the process markedly. The computer soon learns which to use in a particular situation.

Adding up the changes made give $Y = 1.00$ and $X = 2.00$.

Let us assume that an arithmetical error was made and the erroneous result $Y = 1.00$ and $X = 2.50$ was obtained.

On recalculating the residuals the values are found to be $R_1 = .50$ and $R_2 = 1.00$.

Starting out from these values the calculation is shown in Table 3.

Table III

$Y_i = 1.0,\ X_i = 2.50$	R_1	R_2
	0.50	1.00
$\Delta X = -.50$	0.00	0.00

Therefore $Y = 1$ and $X = 2.0$.

The advantage of this procedure over going back and locating the error is obvious.

CONVERGENCE OF METHOD

The question arises: does this method work for every set of linear equations? The answer is no and yes. There are many sets of equations for which the method will not work; but such sets can be recognized and by a simple device brought into a form for which the method will work.

The set of equations

$$4X - 5Y = 3$$
$$2X + 4Y = 8$$

which, like the previous set, has the solution $X = 2$ and $Y = 1$ is a case where the method breaks down. The residuals do not become smaller as we apply more operations. However, if we multiply each equation by the coefficient of its first term and add them, and then multiply by the coefficient of the second term and add two new equations are obtained, namely

$$5X - 3Y = 7$$

$$-12X + 41Y = 17$$

For these equations the method works.

The difference between the two sets is simple. In the first set the coefficient of Y was the largest in both equations while in the second set the coefficient of X was larger in one than that of Y in the other.

The condition for the applicability of the relaxation method is that in each equation a different unknown has the largest coefficient. Such a set is called "well conditioned". Any set can be transformed into a well conditioned set by the device used above.

DIFFERENTIAL EQUATIONS

Perhaps the most important use of relaxation methods has been for solving differential equations. The differential equation is approximated by a set of algebraic equations obtained by replacing derivatives by finite differences.

The basic definition of a derivative found in all calculus books is

$$\frac{dY}{dX} = \underset{\Delta X \to 0}{\text{limit}} \frac{Y(X+\Delta X) - Y(X)}{\Delta X}$$

It took mathematicians more than a hundred years to clearly understand the implications of limit $\Delta X \to 0$. In considering finite differences this difficulty is avoided; ΔX is taken as a small but finite quantity. However, the question as to when the solution of the finite difference equation approaches that of the differential equation is also an extremely difficult problem which is being actively attacked at present. However, we will not discuss this problem here.

Relaxation methods are mostly employed to solve partial differential equations in two variables over regions which are not geometrically simple. It is extremely difficult to apply analytical methods to irregular configurations. Such problems may correspond to systems of several hundred simultaneous equations, yet they can be solved by a single computer in a reasonable time.

Actually the set of simultaneous equations are never written down explicitly. The work is done on a diagram such as Fig. 1a. The residuals are calculated directly and instead of an operations table a "relaxation pattern" is used which shows how a unit change of the unknown at one point changes the residuals. As the difference equations connect points close together a change in one of the unknowns only changes the residuals

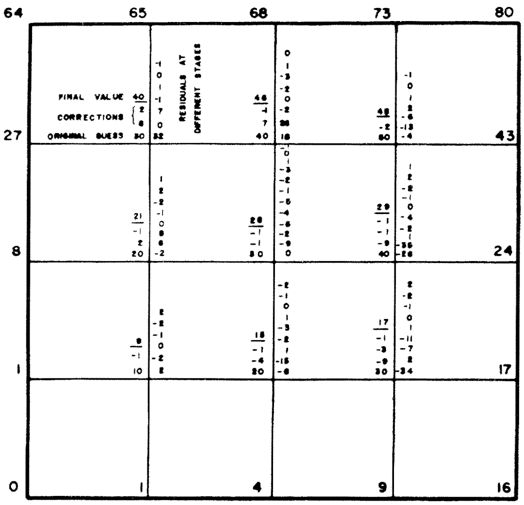

Figure 1a

Details of the relaxation of Laplace's equation with specified boundary conditions in a square. By permission from "Relaxation Methods" by D. N. de G Allen. Copyright 1954, McGraw-Hill Book Company, Inc.

in the immediate vicinity. Therefore, a single relaxation pattern applies everywhere except near a boundary.

The scope of relaxation methods is constantly being extended. For

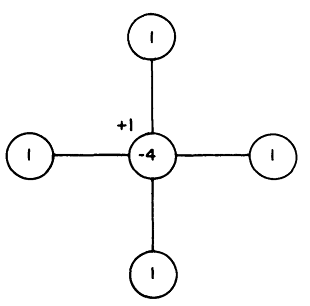

Figure 1b

Relaxation pattern for Laplace's equation showing the change in the residuals at a point and the neighboring points for a unit change in the dependent variable.

example three dimensional problems are being attacked. Allen and Dennis have done this by using isometric projection (Fig. 2) while Tranter has used an ingenious device to reduce three-dimensional problems to two-dimensional ones.

An even more important advance is the application of relaxation methods to initial value problems by Allen and Severn in England and W. P. Reid in the U. S. Initial value problems are one of the basic types of physical

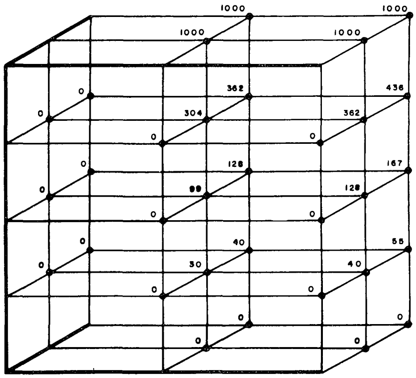

Figure 2

Final solution of Laplace's equation in a cube utilizing isometric projection. By permission from "Relaxation Methods" by D. N. de G Allen. Copyright 1954, McGraw-Hill Book Company, Inc.

problems as opposed to boundary value problems. Simple examples are the motion of a projectile whose initial position and velocity are specified which is a typical initial value problem; while the shape of a beam whose ends are at a specified position is a typical boundary value problem. Previously relaxation methods were applied only to boundary value problems.

LIMITATIONS OF RELAXATION METHODS

In the previous section we have shown the advantages and applications of relaxation methods. Now we will consider its limitations.

Its basic limitation is common to all numerical methods. To obtain a numerical solution of a problem it is necessary to assign numerical values to all the parameters of the problem. Then the solution gives no indication of its dependence on these parameters. To obtain such knowledge it is necessary to solve for a large number of values of the parameters. For a complex problem with many parameters the total amount of work may become prohibitive although a solution for any one set of parameters is straightforward. Furthermore even a large set of numerical solutions gives no guarantee that the behavior will not change drastically outside the range considered.

Another important limitation is that relaxation methods are basically for solving linear equations. This is because for non-linear equations the effect of a change in value of the residual would depend on the value of the unknown. For example, if instead of our first set of equations we had

$$X + 4Y = 6$$

$$2X - Y^2 = 3$$

which has the solution $X = 2$, $Y = 1$ and $Y = -9$, $X = 42$. The operation table is

	ΔR_1	ΔR_2
ΔX	1	2
ΔY	4	$-2Y$

and this must be changed every time Y is changed. Therefore, the amount of work is greatly increased. Furthermore, the convergence problem is much more difficult because of the existence of more than one solution. Relaxation methods are being used for the numerical solution of partial differential equations because there are no markedly superior alternatives.

However, in view of the rapid progress being made in the field, no final statement can be made about the limitations of the method.

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ON THE ORIGIN OF CARTESIAN SYMBOLISM

H. A. Pogorzelski

The aim of this note is to indicate the plausibility that the extensively used exponential symbolism, generally credited to René Descartes, originated from typographical rather than mathematical influences.

The first known advancement of the present-day exponential symbolism is historically traced to *Geometrie* by René Descartes published, in three Books, in 1637. There are definite indications in Book I that the printer was instructed to set the exponents not in the usual superior position, e. g., a^4, which would have been much easier for the printer to handle, but directly over the bases, e. g., $\overset{4}{a}$. This point is substantiated by the fact that in all the exponential occurences, in Book I, the related lines are spread apart, e. g., on page 299, of course, in archaic French:

> "............................ ; Et *aa*,
> ou $\overset{2}{a}$, pour multiplier *a* par soy mesme; Et $\overset{3}{a}$, pour le
> multiplier encore vne fois par *a*, & ainsi a l'infini;"

Moreover, the fact that the bases are printed in conventional italic, i. e., slanted letters, strongly suggests that the printer set the exponents not directly over the bases as instructed but directly in line with the italic bases, as shown in the above extract, thus giving the impression that an exponent was meant to be set a little to the right of its base. On the other hand, in Book III, the exponents are uniformly set as superiors to their respective bases and moreover the related lines are no longer spread apart. This clearly indicates that there must have been some discussion between Descartes and his printer on simplifying the typography used in the first two Books of *Geometrie*, since Book II in particular shows considerable confusion with respect to the printing of exponents. Hence, it is very plausible that the "typographical" difficulties led Descartes to establishing the present-day exponential notation rather than any "mathematical" process.

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A GENERALIZATION OF THE LEGENDRE TRANSFORMATION

R. G. Buschman and N. J. Damaskos

In the past few years several papers concerned with the finite Legendre transformation

$$(1) \qquad f_n = \wp_n\{F(x)\} = \int_{-1}^{+1} F(x)P_n(x)dx$$

have appeared. Applications have been given by Churchill [2] and Trantner [7]. Several of the operational properties have been investigated by Churchill [2] and a convolution theorem was given by Churchill and Dolph [3]. In this paper we consider one method by which such a transformation can be generalized and we derive several properties of this generalization. Consider

$$(2) \qquad f_n(s) = \wp_{n,s}\{F(x)\} = \int_{-1}^{+1} F(x)P_n(sx)dx \ ,$$

which reduces to (1) for $s = 1$, i.e. $\wp_{n,1}\{F(x)\} = \wp_n\{F(x)\}$ or $f_n(1) = f_n$ so that properties of the transformation (1) can be obtained from those of the transformation (2). We will denote the generalized Legendre transform of $F(x)$ by $f_n(s)$ as indicated by (2).

First we state two theorems of Damaskos [4] for which $s = 1$ gives some of the results of Churchill [2].

Theorem 1. If $\wp_{n,s}\{F(x)\} = f_n(s)$ and $\wp_{n,s}\{G(x)\} = g_n(s)$ exist, then $\wp_{n,s}\{AF(x)+BG(x)\} = Af_n(s)+Bg_n(s)$, i.e. the transformation is linear.

Theorem 2. If $\wp_{n,s}\{F(x)\} = f_n(s)$ exists, then $\wp_{n,s}\{F(-x)\} = (-1)^n f_n(s)$.

These two theorems as well as the two following can be proved by methods analogous to those of Churchill [2] and reduce to the same results for $s = 1$ if we take in addition $G(-1) = 0$ in theorem 3.

Theorem 3. If $F(x)$ is sectionally continuous then

$$(3) \qquad \wp_{n,s}\left\{\int_a^x F(t)dt\right\} = \left[\int_a^1 F(x)dx + (-1)^n\int_a^{-1} F(x)dx\right]\frac{P_{n+1}(s)-P_{n-1}(s)}{(2n+1)s} - \frac{f_{n+1}(s)-f_{n-1}(s)}{(2n+1)s} \ ,$$

where if $n = 0$, $P_{-1}(s)$ and $f_{-1}(s)$ are both taken to be zero.

If $a = -1$, then the result of theorem 3 reduces to

$$\wp_{n,s}\left\{\int_{-1}^{x} F(t)dt\right\} = f_0(s)\frac{P_{n+1}(s)-P_{n-1}(s)}{(2n+1)s} - \frac{f_{n+1}(s)-f_{n-1}(s)}{(2n+1)s} . \tag{4}$$

Then, since $P_n(1) = 1$ and $f_n(1) = f_n$ for $s = 1$, we have

$$\wp_{n,1}\left\{\int_{-1}^{x} F(t)dt\right\} = \frac{f_{n-1}-f_{n+1}}{2n+1} . \tag{5}$$

Theorem 4. If $G(x)$ is continuous and $G'(x)$ is sectionally continuous, then

$$\wp_{2m,s}\{G'(x)\} = [G(1)-G(-1)]P_{2m}(s) - s\sum_{k=0}^{m}(4k-1)g_{2k-1}(s) \tag{6e}$$

$$\wp_{2m+1,s}\{G'(x)\} = [G(1)+G(-1)]P_{2m+1}(s) - s\sum_{k=0}^{m}(4k+1)g_{2k}(s) \tag{6o}$$

where we take $g_{-1}(s) \equiv 0$ for the case $2m = 0$.

Using the notation for the Riemann-Liouville fractional integral [6]

$$\Re_\mu\{F(t); x\} = \frac{1}{\Gamma(\mu)}\int_0^x F(t)(x-t)^{\mu-1}dt ,$$

since also

$$\int_0^x\int_0^{t_n}\cdots\int_0^{t_2} F(t_1)dt_1 \cdots dt_{n-1}dt_n = \frac{1}{(n-1)!}\int_0^x F(t)(x-t)^{n-1}dt ,$$

we have

$$\int_0^x\int_0^{t_n}\cdots\int_0^{t_2} F(t_1)dt_1 \cdots dt_{n-1}dt_n = \Re_n\{F(t); x\} .$$

If the integral in (2) is integrated by parts once

$$\wp_{n,s}\{F(x)\} = P_n(s)[\Re_1\{F(x); 1\}-(-1)^n\Re_1\{F(x); -1\}]-s\int_{-1}^{+1}\Re_1\{F(t); x\}P_n'(sx)dx ;$$

if n times,

$$\wp_{n,s}\{F(x)\} = \sum_{k=0}^{n}(-1)^k s^k P_n^{(k)}(s)[\Re_{k+1}\{F(x); 1\}-(-1)^{n-k}\Re_{k+1}\{F(x); -1\}] . \tag{7}$$

This shows that the transformation can be written as a finite sum involving powers and Legendre polynomials and merely fractional integrals of original function for which tables exist [6, 13.1]. From this we can obtain further expressions, for example from [6, 13.1(1)] we get

$$(8)\quad \wp_{n,s}\{F(ax)\}$$

$$= \sum_{k=0}^{n} (-1)^k s^k P_n^{(k)}(s) a^{-k-1}[\Re_{k+1}\{F(x); a\} - (-1)^{n-k}\Re_{k+1}\{F(x); -a\}] ,$$

similarly as in [4]

$$(9)\quad \wp_{n,s}\{F(x+b)\}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} s^k P_n^{(k)}(s) \left[\int_b^{b+1} F(u)(b+1-u)^k du - (-1)^{n-k}\int_b^{b-1} F(u)(b-1-u)^k du\right]$$

and by combining these last two relations

$$(10)\quad \wp_{n,s}\{F(ax+b)\}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{ak!} \frac{s^k}{a^k} P_n^{(k)}(s) \left[\int_b^{b+a} F(u)(b+a-u)^k du - (-1)^{n-k}\int_b^{b-a} F(u)(b-a-u)^k du\right] .$$

An expression for the derivative of the transform is easy to obtain and is a useful relation [4].

Theorem 5.

$$f_n^{(r)}(s) = \begin{cases} \displaystyle\int_{-1}^{+1} x^r F(x) P_n^{(r)}(sx) dx , & r \leqq n ; \\ 0 & r > n . \end{cases}$$

($P_n^{(r)}(sx)$ denotes $d^r P_n(sx)/d(sx)^r$.)

From this we will use the notation

$$f_n^{(r)} = f_n^{(r)}(1) = \int_{-1}^{+1} x^r F(x) P_n^{(r)}(x)\, dx , \qquad r \leqq n ; = 0 ,\ r > n .$$

Integration of the right hand member in Theorem 5 by parts and collection of terms produces

$$(11) \int_{-1}^{+1} \frac{d}{dx}[x^r F(x)] P_n^{(r-1)}(sx) dx = P_n^{(r-1)}(s)[F(1) + (-1)^n F(-1)] - s f_n^{(r)}(s) ,$$

which for $r = 1$ can be written

$$(12) \qquad \wp_{n,s}\{xF'(x)\} = P_n(s)[F(1)+(-1)^n F(-1)] - sf_n'(s) - f_n(s) .$$

From a second such integration by parts, we have the result

$$(13) \qquad \int_{-1}^{+1} \frac{d^2}{dx^2}[x^r F(x)] P_n^{(r-2)}(sx)\,dx = P_n^{(r-2)}(s)[F'(1) - (-1)^n F'(-1)]$$

$$+ [rP_n^{(r-2)}(s) - sP_n^{(r-1)}(s)][F(1)+(-1)^n F(-1)] + s^2 f_n^{(r)}(s) .$$

Equations (12) and (13) for $r = 2$ can be combined to yield

$$(14) \qquad \wp_{n,s}\{x^2 F''(x)\} = P_n(s)[F'(1) - (-1)^n F'(-1)]$$

$$- [2P_n(s) + sP_n'(s)][F(1) + (-1)^n F(-1)]$$

$$+ s^2 f_n''(s) + 4sf_n'(s) + 2f_n(s) .$$

Another source of available relations is the set of recursion formulas for the Legendre polynomials as for example in [5; 10.10]. For example, starting from

$$(2n+1)P_n(x) = P'_{(n+1)}(x) - P'_{(n-1)}(x) ,$$

replacing x by sx, multiplying by $xF(x)$, integrating, and making use of theorem 5 for $r = 1$, we obtain

$$(15) \qquad \wp_{n,s}\{xF(x)\} = \frac{1}{2n+1}[f'_{n+1}(s) - f'_{n-1}(s)] .$$

An alternate form of (15) can be obtained from

$$(2n+1)\, x\, P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

by replacing x by sx, multiplying by $F(x)$ and integrating, that is

$$(16) \qquad \wp_{n,s}\{xF(x)\} = \frac{1}{(2n+1)s}[(n+1)f_{n+1}(s) + nf_{n-1}(s)] .$$

Certain expressions involving the derivatives of transforms can be altered by use of the relation

$$(17) \qquad (n+1)f_{n+1}(s) + nf_{n-1}(s) = s[f'_{n+1}(s) - f'_{n-1}(s)] ,$$

which follows directly from (15) and (16). A special case of (16) is

$$\wp_n\{xF(x)\} = \frac{n+1}{2n+1} f_{n+1} + \frac{n}{2n+1} f_{n-1} .$$

If in the derivation of (16) we multiply by $xF(x)$ instead of $F(x)$ and use (16) in the simplification we obtain

$$(18)\qquad \wp_{n,s}\{x^2F(x)\} = \frac{(n+1)}{(2n+1)(2n+3)s^2}[(n+2)f_{n+2}(s)+(n+1)f_n(s)] + \frac{n}{(2n+1)(2n-1)s^2}[nf_n(s)+(n-1)f_{n-2}(s)]$$

which for $s = 1$ becomes

$$\wp_n\{x^2F(x)\} = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}f_{n+2} + \left[\frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{(2n+1)(2n-1)}\right] f_n + \frac{n(n-1)}{(2n+1)(2n-1)}f_{n-2} \;.$$

Similarly, expressions for the transforms of $x^kF(x)$ for $k > 2$ can be derived.

If we begin with

$$P_n'(x) = xP_{n+1}'(x) - (n+1)P_{n+1}(x)$$

then by similar methods and the use of theorem 5 for $r = 1$, we have

$$(19)\qquad \int_{-1}^{+1} F(x)P_n'(sx)dx = sf_{n+1}'(s) - (n+1)f_{n+1}(s)\;,$$

a useful formula. Integrating the left hand member by parts, the result can be expressed as

$$(20)\quad \wp_{n,s}\{F'(x)\} = P_n(s)[F(1)-(-1)^nF(-1)] + (n+1)sf_{n+1}(s) - s^2f_{n+1}'(s)\;.$$

A slightly different formula for the derivative can be obtained from

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$$

or by combining (17) with (20). In either case

$$(21)\qquad \wp_{n,s}\{F'(x)\} = P_n(s)[F(1)-(-1)^nF(-1)] - s^2f_{n-1}'(s) - nsf_{n-1}(s)\;.$$

Similar manipulations, starting with Legendre's differential equation, produce

$$(22)\ \wp_{n,s}\{F''(x)\} = P_n(s)[F'(1)-(-1)^nF'(-1)] - sP_n(s)[F(1)+(-1)^nF(-1)] + s^4f_n''(s) + 2s^3f_n'(s) - n(n+1)s^2f_n(s)\;.$$

Another relation is

(23) $\wp_{n,s}\{xF'(x)\} = P_n(s)[F(1)-(-1)^nF(-1)]-\frac{n+2}{2n+1}f'_{n+1}(s)-\frac{n-1}{2n+1}f'_{n-1}(s)$.

Further cases of the type $\wp_{n,s}\{x^rF^{(m)}(x)\}$ could also be obtained by these methods, but an explicit general formula for this seems difficult.

Some miscellaneous results are

$$\wp_{n,s}\{F(x)\} = 2^{-n}\sum_{m=0}^{[n/2]}(-1)^m\binom{n}{m}\binom{2n-2m}{n}s^{n-2m}\int_{-1}^{+1}x^{n-2m}F(x)\,dx$$

and using $F(x) = x^r$ in this formula gives

$$\wp_{n,s}\{x^r\} = 2^{-n}\sum_{m=0}^{[n/2]}(-1)^m\binom{n}{m}\binom{2n-2m}{n}s^{n-2m}\left[\frac{1+(-1)^{n+r}}{n-2m+r+1}\right]$$

and then in formula 7 gives

$$\wp_{n,s}\{x^r\} = \sum_{k=0}^{n}(-1)^k[1+(-1)^{n+r}]\frac{r!}{(r+k+1)!}s^kP_n^{(k)}(s) .$$

Some further transforms are

$$\begin{aligned}\wp_{n,s}\{P_m(x)\} &= 0 , && n < m \text{ or } n > m \text{ and } (n+m) \text{ odd};\\ &= \frac{2}{2n+1}s^n && n = m ;\\ &= p_n(s) , && n > m \text{ and } (n+m) \text{ even} ,\end{aligned}$$

where $p_n(s)$ is a polynomial in s of degree n. One special case is

$$\wp_{n,s}\{P_{n-2}(x)\} = s^n - s^{n-2} .$$

Also we have

$$\begin{aligned}\wp_{n,s}\left\{\frac{P_{n+1}(x)}{x}\right\} &= \frac{2}{n+1} , && n \text{ even};\\ &= 0 , && n \text{ odd};\\ \wp_{n,s}\left\{\frac{P_{n-1}(x)}{x}\right\} &= \frac{2}{n}s^n , && n \text{ odd};\\ &= \frac{2}{n}(1-s^n) , && n \text{ even} .\end{aligned}$$

Substitution theorems of the type which yield expressions for the transform of $k(x)F[g(x)]$ and of the type for the function which has the transform $k(x)f_n[g(s)]$ can be obtained by methods similar to those of [1].

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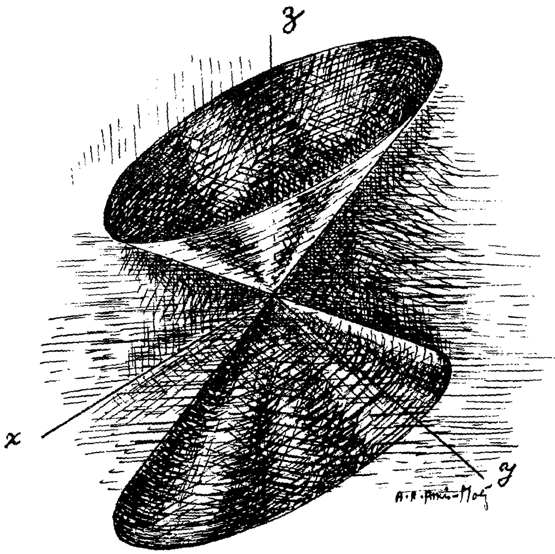
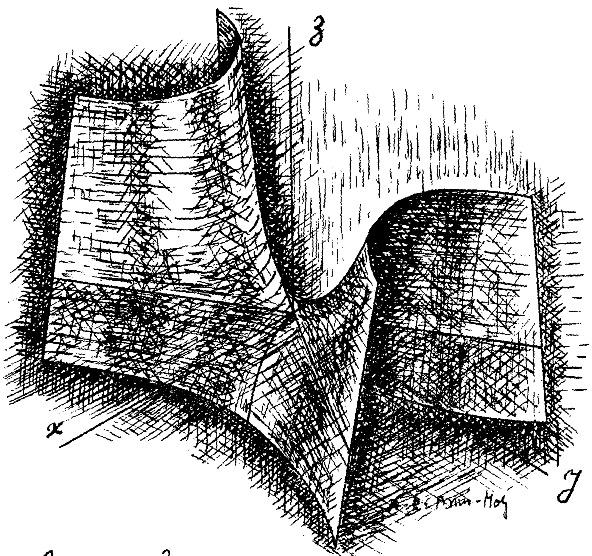


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + 2cz = 0.$$

UNIQUE DIFFERENTIAL RATIOS AND CURVILINEAR COORDINATES

H. Randolph Pyle

Introduction.

Let (x, y) be the rectangular coordinates of a point in one plane and (u, v) the coordinates of a corresponding point in a second plane, and let them be related by the transformation $u = \phi(x, y)$, $v = \psi(x, y)$. If $ds^2 = dx^2 + dy^2$ and $ds'^2 = du^2 + dv^2$, the condition for the conformal mapping of one plane on the other is that $ds'^2 = t^2 ds^2$. This relation is shown in the classical theory of functions of a complex variable to be equivalent to the Cauchy-Riemann equations.

If $dw = du + idv$ and $dz = dx + idy$, the ratio of the differentials $\frac{dw}{dz}$ turns out to be unique at a point when and only when the Cauchy-Riemann equations are satisfied, i. e., the conditions for conformal mapping and for a unique ratio of these differentials are the same.

When the coordinate systems are not rectangular the metrics become $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$, $ds'^2 = E'du^2 + 2F'dudv + G'dv^2$. Now when we impose the condition for conformal mapping, $ds'^2 = t^2 ds^2$, the relations between the partial derivatives of ϕ and ψ are still linear but more complex than the Cauchy-Riemann equations [1].

It is the purpose of this paper to show that if we write

$$dw = \frac{E'du + F'dv + iH'dv}{\sqrt{E'}} \quad \text{and} \quad dz = \frac{Edx + Fdy + iHdy}{\sqrt{E}},$$

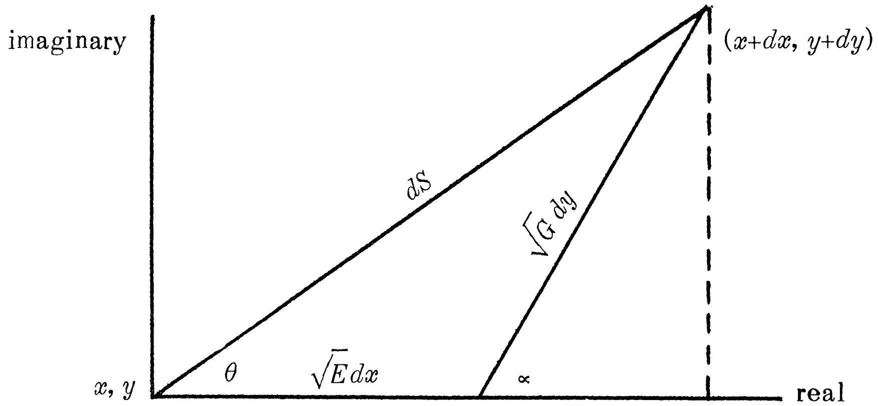
the necessary and sufficient conditions for the uniqueness of the differential ratio $\frac{dw}{dz}$ are the same as those for conformal mapping.

Then we shall show how these relations can be used to solve a considerable class of partial differential equations of the second order.

The differential ratio.

Suppose that we are given a surface S with the metric $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$. We shall join the points $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ by a geodesic curve on the surface. The length of arc is ds. If, with the tangent to this geodesic, we form an infinitesimal triangle in the tangent plane to the surface, with one side along the tangent to the x-coordinate curve through P and the other along the projection of the tangent to the y-coordinate curve through Q, the lengths of the arcs are $\sqrt{E}dx$ and $\sqrt{G}dy$ respectively [2 : 54]. If we call the angle between these coordinate curves α, $\cos\alpha = F/\sqrt{EG}$, $\sin\alpha = H/\sqrt{EG}$, $H^2 = EG - F^2$.

Now if we form a local coordinate system with the axis of reals tangent to the x-coordinate curve and the axis of imaginaries perpendicular to it in



the tangent plane to the surface, the projection of ds on the real axis is $\sqrt{E}dx+\sqrt{G}dy\cos\propto\ =\ \sqrt{E}dx+F/\sqrt{E}dy\ =\ (Edx+Fdy)/\sqrt{E}$. Its projection on the axis of imaginaries is $\sqrt{G}dy\sin\propto\ =Hdy/\sqrt{E}$. If we let $dz\ =\ (Edx+Fdy+iHdy)/\sqrt{E}$ then $ds\ =\ |dz|$. In general dz is not an exact differential. It reduces to the form $dz\ =\ dx+idy$ when the coordinate system is isometric.

Now let S' be a second surface with metric $ds'^2\ =\ E'du^2+2F'dudv+G'dv^2$. The point (u,v) on S' is related to the point (x,y) on S by the transformation $u\ =\ \phi(x,y)$, $v\ =\ \psi(x,y)$. We shall assume that the partial derivatives of ϕ and ψ exist and are continuous, and that the Jacobian does not vanish. If we let $dw\ =\ (E'du+F'dv+iH'dv)/\sqrt{E'}$, we are interested in finding when the ratio of the differentials dw/dz is unique at a given point.

The computation is simplified by a change of notation. Let $E\ =\ Ha$, $F=Hb$, $G=Hc$, so that $ac-b^2\ =\ 1$, with similar substitutions for the primed terms. Then

$$dz\ =\ \sqrt{\frac{H}{a}}(adx+bdy+idy)\ ,\qquad dw\ =\ \sqrt{\frac{H'}{a'}}(a'du+b'dv+idv)\ .$$

Now let $dw\ =\ (\lambda'+i\mu')dz$. If we let $\lambda\ =\ \sqrt{Ha'/H'a}\,\lambda'$, $\mu\ =\ \sqrt{Ha'/H'a}\ \mu'$, we have $a'du+b'dv+idv\ =\ (\lambda+iu)(adx+bdy+idy)$. Equating real and imaginary parts gives

$$a'du+b'dv\ =\ \lambda(adx+bdy)-\mu dy$$
$$dv\ =\ \mu(adx+bdy)+\lambda dy$$

Substituting $du\ =\ \phi_x dx+\phi_y dy$, $dv\ =\ \psi_x dx+\psi_y dy$, and equating the coefficients of dx and dy, we have

$$\text{1a)}\quad \begin{aligned} a'\phi_x+b'\psi_x &= \lambda a \\ a'\phi_y+b'\psi_y &= \lambda b-\mu \end{aligned}\qquad\qquad \text{1b)}\quad \begin{aligned} \psi_x &= \mu a \\ \psi_y &= \mu b+\lambda \end{aligned}$$

From (1b) we get

$$a\psi_y - b\psi_x = \lambda a$$

$$c\psi_x - b\psi_y = \mu(ac - b^2) - \lambda b = \mu - \lambda b$$

or

$$a'\phi_x + b'\psi_x = a\psi_y - b\psi_x$$

2)

$$a'\phi_y + b'\psi_y = b\psi_y - c\psi_x$$

These are the necessary conditions for dw/dz to be unique at a point. They are also the necessary and sufficient conditions for conformal mapping, i. e., $ds'^2 = t^2 ds^2$ [1].

Since $\lambda = \sqrt{\dfrac{Ha'}{H'a}}\lambda' = \dfrac{a'\phi_x + b'\psi_x}{a}$, $\mu = \sqrt{\dfrac{Ha'}{H'a}}\mu' = \dfrac{\psi_x}{a}$

$$\frac{dw}{dz} = \lambda' + i\mu' = \sqrt{\frac{H'}{Haa'}}(a'\phi_x + b'\psi_x + i\psi_x)$$

or

$$\frac{dw}{dz} = \frac{E'\phi_x + F'\psi_x + iH'\psi_x}{\sqrt{EE'}}$$

We shall now show that the equations (2) are sufficient to make the differential ratio unique.

$$\frac{dw}{dz} = \frac{\sqrt{\dfrac{H'}{a'}}\,(a'\phi_x + b'\psi_x)dx + (a'\phi_y + b'\psi_y)dy + i(\psi_x dx + \psi_y dy)}{\sqrt{\dfrac{H}{a}}\,(adx + bdy + idy)}$$

Using equations (2) we may write the numerator in the form

$$\sqrt{\frac{H'}{a'}}[(a\psi_y - b\psi_x)dx + (b\psi_y - c\psi_x)dy + i(\psi_x dx + \psi_y dy)]$$

$$= \sqrt{\frac{H'}{a'}}[\psi_y(adx + bdy + idy) - \psi_x(bdx + cdy - idx)]$$

But

$$(b-i)dx + cdy = \frac{a(b-i)dx + acdy}{a} = \frac{(b-i)adx + (b^2+1)dy}{a}$$

$$= (b-i)\left[\frac{adx + (b+i)dy}{a}\right]$$

so that

$$\frac{dw}{dz} = \sqrt{\frac{H'a}{Ha'}}\left[\frac{(\psi_y - \frac{(b-i)}{a}\psi_x)\{adx+(b+i)dy\}}{adx+(b+i)dy}\right]$$

$$\frac{dw}{dz} = \sqrt{\frac{H'a}{Ha'}}\left[\frac{a\psi_y - b\psi_x + i\psi_x}{a}\right] = \sqrt{\frac{H'}{Haa'}}\,(a'\phi_x + b'\psi_x + i\psi_x)$$

and the conditions are sufficient.

Thus the necessary and sufficient conditions for (dw/dz) to be unique are the same as those that make $ds'^2 = t^2 ds^2$.

An integrating factor for dz.

We have said that in general dz is not an exact differential. We shall now consider the possibility of finding an integrating factor for it.

If we choose an isometric coordinate system for S', $a' = c' = 1$, $b' = 0$, $H' = 1$. Then $dw = du + idv$ is an exact differential. If now $dw = (\lambda' + i\mu')dz$, $\lambda' + i\mu'$ represents an integrating factor for dz. Under these conditions equations (1a, b) reduce to $\phi_x = \lambda a$, $\phi_y = \lambda b - \mu$, $\psi_x = \mu a$, $\psi_y = \mu b + \lambda$. If we assume that a, b, λ, μ have suitable partial derivatives, we have

$$\frac{\partial}{\partial y}(\lambda a) = \frac{\partial}{\partial x}(\lambda b - \mu)\,, \qquad \frac{\partial}{\partial y}(\mu a) = \frac{\partial}{\partial x}(\mu b + \lambda)\,.$$

$$\lambda(a_y - b_x) + a\lambda_y = b\lambda_x - \mu_x$$

$$\mu(a_y - b_x) + a\mu_y = b\mu_x + \lambda_x$$

These equations have an easy solution if we set $\mu = 0$, i.e., we impose the condition that (dw/dz) be real. Then $\lambda_x = 0$, and λ is a function of y only. Now $\lambda(a_y - b_x) + a\lambda_y = 0$, or $\dfrac{\lambda_y}{\lambda} = \dfrac{b_x - a_y}{a}$ and $\lambda = e^{\int \frac{b_x - a_y}{a} dy}$. This imposes a necessary condition on the coefficients a and b, namely that $\dfrac{b_x - a_y}{a}$ be a function of y only.

We shall see that a large class of metrics, including those of surfaces of revolution and the non-euclidean geometries meet this condition.

Since

$$d\phi = \lambda(adx + bdy)\,, \qquad d\psi = \lambda dy\,,$$

$$\phi = \lambda\int^x adx\,, \qquad \psi = \int^y \lambda dy\,.$$

Now $\phi_x = \lambda a$, $\phi_y = \lambda\int^x a_y dx + \lambda_y \int^x adx = \int^x(\lambda a_y + \lambda_y a)dx$. But $\lambda_y = \lambda\dfrac{(b_x - a_y)}{a}$, so that $\phi_y = \int^x[\lambda a_y + \lambda(b_x - a_y)]dx$ and $\phi_y = \lambda\int^x b_x dx = \lambda b$ as is required.

This gives

3) $$w = \phi + i\psi = \lambda\int^x adx + i\int^y \lambda dy\,,$$

when $\lambda = e^{\int \frac{b_x - a_y}{a} dy}$ is a function of y only.

The generalized Laplace equation.

When $dw = d\phi + id\psi$, $a' = c' = 1$, $b' = 0$, $H' = 1$ and equations (2) become

$$\phi_x = a\psi_y - b\psi_x \,, \quad \phi_y = b\psi_y - c\psi_x \,,$$

and

$$\frac{\partial}{\partial y}(a\psi_y - b\psi_x) = \frac{\partial}{\partial x}(b\psi_y - c\psi_x)$$

or

$$c\psi_{xx} + a\psi_{yy} - 2b\psi_{xy} + (c_x - b_y)\psi_x - (b_x - a_y)\psi_y = 0 \,.$$

Since $b^2 - ac = -1$, this is a differential equation of elliptic type [3:108]. Both ϕ and ψ, when they are defined as in equation (3), are solutions of this equation. This may be verified by substitution.

Examples.

For brevity we shall give the results only.

1) Polar coordinates.

$$ds^2 = dr^2 + r^2 d\theta^2 \,, \qquad \lambda = 1 \,,$$

$$dz = \frac{1}{r} dr + id\theta \,, \qquad z = \ln r + i\theta$$

is a solution of the Laplace equation

$$r^2\psi_{rr} + \psi_{\theta\theta} + r\psi_r = 0 \,.$$

2) Any surface of revolution has the metric

$$ds^2 = (1 + f'^2)dx^2 + x^2 dy^2$$

where $f = f(x)$ only. Now

$$\lambda = 1 \,, \qquad z = \int^x \frac{\sqrt{1+f'^2}}{x} dx + iy \,.$$

The Laplace equation is

$$x^2\psi_{xx} + (1+f'^2)\psi_{yy} + \frac{x(1+f'^2 - xf'f'')}{1+f'^2}\psi_x = 0 \,.$$

3) Sphere.

$$ds^2 = r^2 dx^2 + r^2 \sin^2 x dy^2 \,, \qquad \lambda = 1 \,, \qquad z = \ln \tan \frac{x}{2} + iy \,.$$

The Mercator projection is

$$\phi = -\ln \tan (x/2) \,, \qquad \psi = y \,,$$

so that our forms give the Mercator projection formulas for this metric.
4) Elliptic geometry.

$$ds^2 = \frac{k^2[(k^2+y^2)dx^2 + (k^2+x^2)dy^2 - 2xydxdy]}{(x^2+y^2+k^2)^2}$$

Here

$$\frac{b_x - a_y}{a} = \frac{-2y}{k^2+y^2}, \qquad \lambda = \frac{1}{k^2+y^2}$$

$$\phi = \frac{1}{k}\sinh^{-1}\frac{x}{\sqrt{k^2+y^2}}, \qquad \psi = \frac{1}{k}\tan^{-1}\frac{y}{k}$$

The Laplace equation is

$$(k^2+x^2)\psi_{xx} + (k^2+y^2)\psi_{yy} + 2xy\psi_{xy} + 2x\psi_x + 2y\psi_y = 0\ .$$

5) Hyperbolic geometry.

If we replace k by ik in the elliptic geometry we get the hyperbolic case. Carrying out this substitution gives

$$\phi = \frac{1}{k}\sin^{-1}\frac{x}{\sqrt{k^2-y^2}}, \qquad \psi = \tanh^{-1}\frac{y}{k}\ .$$

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COMPLETION OF SEMINORMED SPACES AND THE DANIELL PROCESS OF EXTENDING AN INTEGRAL

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Introduction. The Daniell extension is an abstract formulation of the process involved in obtaining the Lebesgue integral from the Riemann integral. The starting point is a postulated "elementary integral" on a corresponding "elementary" class L of functions which is extended to a larger class $\overline{L}$. In Stone's [2] treatment $\overline{L}$ is obtained as the closure of L under a suitable norm N on a vector space of functions containing L, and the extended integral $\overline{\int}$ is defined by a formula containing N. This method of obtaining $\overline{L}$ is in keeping with the well-known fact [3, 8] that the class of Lebesgue integrable functions is the completion of the vector space of Riemann integrable functions, and the Lebesgue integral the extension to the completion of the linear functional which is the Riemann integral. This fact is seldom used to define the Lebesgue integral, (however, see [3]) because the process of completing a vector space of functions does not yield directly elements of the new space that we can call "functions."

In this expository work we develop the Daniell-Stone theory from a point of view that differs from that of these authors in that it places greater emphasis on the nature of the various vector spaces of functions involved. We believe that the resulting exposition is simpler and more natural. We give no special status to elementary functions or integrals. Instead, with Bourbaki [5], we define a *measure* as a pair $\mu = (L, \int)$ satisfying Stone's axioms [2] and call it *complete* if L is complete under $N_1(f) = \int|f|$. In Section 4 we complete the seminormed space L of an arbitrary measure μ in the usual way by means of Cauchy sequences into its completion $\tilde{L}$, and extend the functional $\int$ to obtain $\tilde{\int}$. Then we resolve the difficulty, mentioned above, of considering $\tilde{L}$ as a space of functions. We do this by means of a natural process of assigning functions to elements of $\tilde{L}$ to form a new vector space $\overline{L}$. $\overline{\int}$ is easily defined. In Theorem 10, we deduce that $(\overline{L}, \overline{\int})$ is indeed a complete measure from the well-known topological properties of $(\tilde{L}, \tilde{\int})$.

In Sections 2 and 3 we have already exhibited the essential properties of $(\overline{L}, \overline{\int})$, i. e. the Lebesgue theory. In Section 2 we present the part which depends on Stone's axioms alone, and is valid for any (complete or incomplete) measure. The reader will recognize here theorems that are elsewhere presented at the $(\overline{L}, \overline{\int})$ level, their usual proofs unnecessarily complicated by constant references to the passage from $(L, \int)$ to $(\overline{L}, \overline{\int})$. In Section 3, the remainder of the theory, dependent in addition upon the axiom of completeness, is given. Again the cumbersome references to the completion process are eliminated. They are replaced in the proofs by the systematic application of completeness itself. The auxiliary material on measurability sketched in Section 1 has been essentially reproduced from Stone [1, 2].

1. *Baire closure*. Through this paper only real (finite) numbers are considered as limits of sequences of real numbers, except in the case of the symbol $\overset{\infty}{\underset{i=1}{\uparrow}} a_i$, which stands for the l. u. b. (perhaps infinite) of an increasing sequence $\{a_i\}_i$ of real numbers. Since another way of expressing this special type of limits is $\sum_{i=1}^{\infty} b_i$, $b_i \geqq 0$, the latter carries a similar interpretation. Similarly for $\overset{\infty}{\underset{i=1}{\downarrow}} a_i$ and $\sum_{i=1}^{\infty} b_i$, $b_i \leqq 0$. Correspondingly, all functions considered are real (finite) valued except those described in terms of $\overset{\infty}{\underset{i=1}{\uparrow}} f_i$, $\overset{\infty}{\underset{i=1}{\downarrow}} f_i$ and the corresponding summations. As usual $f^+ \geqq 0$ and $f^- \leqq 0$ represent respectively the positive and negative part of a function f.

Let S be a set and L a vector lattice of functions on S. The *Baire closure* L_B of L is defined as the smallest class of functions on S closed under sequential pointwise limits. L_B can be proved to be a vector lattice (Loomis [4], p. 32, 12H). In the same way one establishes that if L satisfies

$$f \in L \Rightarrow f \wedge 1 \in L \tag{1}$$

so does L_B.

For vector lattices (1) is equivalent to

$$f \in L\,, \qquad a > 0 \Rightarrow f \wedge a \in L \tag{2}$$

and also to

$$f \in L\,, \qquad a < 0 \Rightarrow f \vee a \in L\ . \tag{3}$$

Theorem 1. If L satisfies (1), $f_1, \cdots, f_n \in L_B$, ϕ a continuous function on euclidean n-space E^n, vanishing at the origin, then $\phi(f_1 \cdots f_n) \in L_B$.

Proof (Stone [2], p. 341) Let κ be the smallest vector lattice of functions on E^n satisfying (1) and containing the coordinate functions $\phi_i(\lambda_1, \cdots, \lambda_n) = \lambda_i$. Then for $\psi \in \kappa$, $\psi(f_1, \cdots, f_n) \in L_B$. If $k > 0$ and $K = \{(\lambda_1, \cdots, \lambda_n) \mid \sum_{i=1}^{n} |\lambda_i| \leqq k\}$ then $\phi \mid K$ is the uniform (hence pointwise) limit of a sequence $\{\psi_j \mid K\}_j$, $\psi_j \in \kappa$, (Stone [1], p. 172 Theorem 3). If $f_i^k = (f_i \wedge \frac{k}{n}) \vee (\frac{-k}{n}) \in L_B$, $(f_i^k(x), \cdots, f_n^k(x)) \in K$ for every $x \in S$, and $\phi(f_1^k, \cdots, f_n^k) = \lim_j \psi_j(f_1^k, \cdots, f_n^k) \in L_B$, and $\phi(f_1, \cdots, f_n) = \lim_{k\to\infty} \phi(f_1^k, \cdots, f_n^k) \in L_B$.

The conclusion of the above theorem remains valid if the continuous function ϕ is replaced by a Baire function, i. e., in the Baire closure of the class of all continuous functions on E^n. It follows from the theorem that L_B is a real algebra under the usual function multiplication.

2. *Abstract measures*. Let L be a vector lattice of functions on S and $\int$ a positive real linear functional on L. A subset A of S is said to be a *null set* if, for every $\epsilon > 0$ there is an increasing sequence $\{f_i\}_i$, $f_i \in L$ so that its characteristic function $\chi_A \leqq \overset{\infty}{\underset{i=1}{\uparrow}} f_i$ and $\int f_i < \epsilon$ for every i. *Almost everywhere* refers to a property of points valid outside of a null set, (abbreviated: a. e.).

For $p \geqq 1$, we let $L_p = \{f \mid f \in L_B,\ |f|^p \in L\}$ and $N_p(f) = [\int |f|^p]^{(1/p)}$, for $f \in L_p$. It is customary to identify functions of L_p that coincide a. e.. We do not follow this convention. Hence L_p is genuinely a set of functions, not of equivalence classes. Thus N_p is never a norm. It may not be even a seminorm due to the weak hypothesis on $(L, \int)$. For the same reason L_p may fail to be a vector space. However, N_1 is a seminorm on $L \subset L_1$. In case L_p is a vector space and N_p a seminorm, N_p defines a (non-Hausdorff) seminorm topology on L_p. All topological notions relative to this topology will be prefixed with N_p: e. g., N_p-convergence. $\int$ is N_1-continuous on L.

With Bourbaki ([6], p. 114 No. 5) we call $\mu = (L, \int)$ an *abstract measure* of S if L is a vector lattice of functions on S satisfying (1) and $\int$ a positive real linear functional on L so that

$$(4) \qquad 0 = \overset{\infty}{\underset{i=1}{\downarrow}} f_i\,, \quad f_i \in L \Rightarrow 0 = \overset{\infty}{\underset{i=1}{\downarrow}} \int f_i\,.$$

For positive real functionals on vector lattices of functions on S, (4) is equivalent to each of the following:

$$(5) \qquad 0 = \overset{\infty}{\underset{i=1}{\uparrow}} f_i\,, \quad f_i \in L \Rightarrow 0 = \overset{\infty}{\underset{i=1}{\uparrow}} \int f_i$$

$$(6) \qquad f = \overset{\infty}{\underset{i=1}{\uparrow}} f_i\,, \quad f,\ f_i \in L \Rightarrow \int f = \overset{\infty}{\underset{i=1}{\uparrow}} \int f_i\,.$$

$$f = \overset{\infty}{\underset{i=1}{\downarrow}} f_i\,,\ f,\ f_i \in L \Rightarrow \int f_i = \overset{\infty}{\underset{i=1}{\downarrow}} \int f_i\,. \tag{7}$$

$$f \leqq \overset{\infty}{\underset{i=1}{\uparrow}} f_i,\ f,\ f_i \in L \Rightarrow \int f \leqq \overset{\infty}{\underset{i=1}{\uparrow}} \int f_i \tag{8}$$

$$f \geqq \overset{\infty}{\underset{i=1}{\downarrow}} f_i\,,\ f,\ f_i \in L \Rightarrow \int f \geqq \overset{\infty}{\underset{i=1}{\downarrow}} \int f_i \tag{9}$$

$$\sum_{i=1}^{\infty} |f_i| \geqq f\,,\ f,\ f_i \in L \Rightarrow \sum_{i=1}^{\infty} \int |f_i| \geqq \int f \tag{10}$$

Convergence of functions in the seven conditions (4) through (10) is assumed everywhere. Seven new conditions (4′) through (10′) are obtained if a. e. convergence is used instead. (4′) through (10′) are also equivalent and each implies (4) through (10). The converse is proved in Lemma 1.

If $(L, \int)$ is an abstract measure, then in the definition of null set above each f_i may be assumed $\geqq 0$, for $\chi_A \leqq \overset{\infty}{\underset{i=1}{\uparrow}} f_i \vee 0$ and $f_i \vee 0 \leqq \overset{\infty}{\underset{k=1}{\uparrow}} f_k$. For abstract measures also, subsets of null sets are null, denumerable unions of null sets are null, and a function f of L is 0 a. e. if and only if $\int |f| = 0$. To see the latter we note that if $f = 0$ a. e. and n is any positive integer, then $\int |f| \wedge n = 0$, for if $A = \{x | x \in S;\ |f(x)| > 0\}$ and $\epsilon > 0$, there is an increasing sequence $\{f_i\}_i$, $f_i \in L$, so that $\chi_A \leqq \overset{\infty}{\underset{i=1}{\uparrow}} f_i$ and $\int f_i < \frac{\epsilon}{n}$. Since $|f| \wedge n \leqq \overset{\infty}{\underset{i=1}{\uparrow}} nf_i$ then $\int |f| \wedge n < \epsilon$. Thus $|f| = \overset{\infty}{\underset{n=1}{\uparrow}} |f| \wedge n$, and $\int |f| = 0$. This proves the "only if" part. The "if" part is easier.

Lemma 1. For any pair $(L, \int)$, L a vector lattice of functions satisfying (1), $\int$ a positive real linear functional on L, each one of conditions (4) through (10) is equivalent to each one of (4′) through (10′).

Proof. It suffices to show (4) is equivalent to (4′), and, evidently, (4′) implies (4). So let $\overset{\infty}{\underset{i=1}{\downarrow}} f_i = 0$ a. e.. We may assume f_i decreases everywhere, since $g_i = \overset{i}{\underset{k=1}{\wedge}} f_k$ is a. e. equal to f_i and $\{g_i\}_i$ decreases everywhere. We may also assume $f_i \geqq 0$. Then $f = \overset{\infty}{\underset{i=1}{\downarrow}} f_i = 0$ a. e.. If $A = \{x | x \in S;\ |f(x)| > 0\}$, A is null and for $\epsilon > 0$ and each i there is a sequence $k_i^j \geqq 0$, $k_i^j \in L$ so that $\sum_j k_i^j \geqq \chi_A$ and $\sum_{j=1}^{\infty} \int k_i^j < \frac{\epsilon}{i2^i}$. The double series $\sum_{ij} ik_i^j \geqq f$ and $\sum_{ij} i\int k_i^j < \epsilon$. The partial sums of $\sum_{ij} ik_i^j$ form an increasing sequence $\{h_q\}_q$ so that $\overset{\infty}{\underset{k=1}{\uparrow}} h_q \geqq$

$\overset{\infty}{\underset{i=1}{\downarrow}} f_i$. Consequently $\overset{\infty}{\underset{i=0}{\downarrow}} (f_i - h_i) \leqq 0$ and $\overset{\infty}{\underset{i=1}{\downarrow}} \int f_i \leqq \overset{\infty}{\underset{i=1}{\uparrow}} \int h_i$.

In each of several theorems that follow we make a statement that holds simultaneously for each of the sets L and L_p, $p \geqq 1$, and the corresponding N_p (N_1 for L). Consequently we let E stand for any of these sets and N for the corresponding N_p. In general, there is no guarantee that E is a vector space nor that N is a seminorm, facts that usually will appear as part of the hypothesis. E is a subset of some L_p, for either $E = L_p$ or $E = L \subset L_1$. For this reason, although the theorem is stated in terms of E and N, the proof can and will, without loss of generality be given for $E = L_p$ and $N = N_p$.

Theorem 2. If $\mu = (L, \int)$ is an abstract measure, E a vector space and N a seminorm, then $f_i \in E$, $f_i \to 0$ a. e. and $\{f_i\}_i$ an N-Cauchy sequence imply $\{f_i\}_i$ N-converges to 0.

Proof. Let $f_i \in E$, $f_i \to 0$ a. e., $\{f_i\}_i$ an N-Cauchy sequence. Since $E \subset L_p$ for some $p \geqq 1$ and N_p behaves like a seminorm on linear combinations of f_i we may assume $E = L_p$ and $N = N_p$. By passing to a subsequence we may also assume $N(f_{i+1} - f_i) < 1/2^i$. Then

$$(11) \qquad |f_i|^p \leqq \overset{\infty}{\underset{n=1}{\uparrow}} \left[\sum_{k=i}^{n} |f_{k+1} - f_k| \right]^p \quad \text{a. e.}$$

By (8′), $N_p(f_i) \leqq \sum_{n=i}^{\infty} N_p(f_{k+1} - f_k) \overset{i}{\to} 0$.

Corollary 1. Under the hypothesis of Theorem 2 for μ, E and N, f, $f_i \in E$, $f_i \to f$ a. e. and $\{f_i\}_i$ an N-Cauchy sequence imply $f_i \overset{N}{\to} f$.

Corollary 2. If L is a vector lattice of functions on S satisfying (1) and $\int$ a linear positive functional on L, then (4) (hence each of (5) through (10)) is equivalent to

(12) $f_i \in L$, $f_i \to 0$ a. e. and $\{f_i\}_i$ an N_1-Cauchy sequence imply $f_i \overset{N_1}{\to} 0$.

For the remainder of this paper we shall assume, unless the contrary is explicitly stated, that $\mu = (L, \int)$ stands for an abstract measure.

Lemma 2. If $\{f_i\}_i$ is an a. e. increasing sequence of functions of L so that $\{\int f_i\}_i$ is bounded, then $\{f_i\}_i$ is bounded a. e..

Proof. We may assume $f_i \geqq 0$ and that $\{f_i\}_i$ is everywhere increasing. Let $A = \{x \mid x \in S, \overset{\infty}{\underset{i=1}{\uparrow}} f_i(x) = +\infty\}$. For every $n > 0$, $\overset{\infty}{\underset{i=1}{\uparrow}} \frac{1}{n} f_i = +\infty > \chi_A$ and $\overset{\infty}{\underset{i=1}{\uparrow}} \frac{1}{n} \int f_i$ can be made arbitrarily small with n. Hence A is null.

Theorem 3. If E is a vector space, N a seminorm, $\{f_i\}_i$ a sequence of E so that $\sum_{i=1}^{\infty} N(f_{k+1}-f_k) < +\infty$, then $\{f_i\}_i$ converges a. e..

Proof. As before, we may assume $E = L_p$ and $N = N_p$ for some $p \geqq 1$. First let $f_i \geqq 0$.

$$\int\left[\sum_{i=1}^{n} |f_{i+1}-f_i|\right]^p = \left[N_p\left(\sum_{i=1}^{n} |f_{i+1}-f_i|\right)\right]^p \leqq \left[\sum_{i=1}^{\infty} N_p(f_{i+1}-f_i)\right]^p$$

By Lemma 2, $\sum_{i=1}^{\infty} |f_{i+1}-f_i| < +\infty$ a. e.. Hence $f_n = \sum_{i=1}^{n-1}(f_{i+1}-f_i)+f_1$ converges absolutely a. e..

In general, if $\{f_i\}_i$ satisfies the hypothesis so do f_i^+ and f_i^-, hence our theorem.

The Next theorem is essentially a converse of Theorem 2.

Theorem 4. If E is a vector space, N a seminorm and $\sum_{i=1}^{\infty} N(f_i) < +\infty$, then $f_i \to 0$ a. e..

Proof. $\{f_i\}_i$ satisfies the hypothesis of Theorem 3 and converges a. e. to a function f. We show $f = 0$ a. e.. By passing to a subsequence, we may assume $\sum_{i=1}^{\infty} iN_p(f_i) < +\infty$. We may also assume $f_i \geqq 0$. Let $A = \{x | x \in S; \limsup f_i(x) > 0\}$, and $g_{in} = \bigvee_{k=n}^{i} kf_k$, $\uparrow_{i=n}^{\infty} (g_{in})^p \geqq \chi_A$ But $g_{in} \leqq \sum_{k=n}^{i} kf_k$, and

$$\int(g_{in})^p \leqq \left[\sum_{k=n}^{\infty} kN_p(f_k)\right]^p \tag{13}$$

The right of (13) can be made arbitrarily small with n. Hence S is null.

Corollary 3. Under the hypothesis of Theorem 3 for E, N and $\{f_i\}_i$, if $f_i \overset{N}{\to} f \in E$ then $f_i \to f$ a. e..

Proof. By Theorem 3, $\{f_i\}_i$ converges a. e.. By passing to a subsequence we may assume $\sum_{i=1}^{\infty} N(f-f_i) < +\infty$ and, by Theorem 4, $f-f_i \to 0$ a. e..

It follows from Theorem 4 that whenever E is a vector space and N a seminorm we can extract out of every N-Cauchy sequence $\{f_i\}_i$ an a. e. convergent subsequence. Furthermore any two such subsequences converge a. e. to the same function f. Hence we will stretch the meaning of a. e. convergence to say that the entire sequence $\{f_i\}_i$ converges to f, and will use this convention hereafter. It should be noted that, as usual, any N-Cauchy sequence has many a. e. limits, any two limits being equal a. e.. Following our practice, we consider all limits as distinct.

Corollary 4. (a) If E is a vector space and N a seminorm, every N-Cauchy sequence converges a. e.. (b) If f, $f_i \in E$, $\{f_i\}_i$ an N-Cauchy sequence, then $f_i \overset{N}{\to} f$ if and only if $f_i \to f$ a. e..

Proof. (a) follows from the discussion preceding the lemma. The "if" part of (b) follows from Corollary 1 and the "only if" part from Corollary 3. The statement and proof of the whole lemma are, of course, based on the above convention for a. e. convergence.

3. *Enlarged and complete measures.* We say an abstract measure $\mu = (L, \int)$ is *enlarged* if every function on S which coincides a. e. with a function of L, belongs to L. In analogy with L_B (Section 1), we denote by $L_{B'}$ the smallest class of functions on S containing L and closed under a. e. sequential limits. We could prove directly that $L_{B'}$ has the same lattice properties as L_B but this follows from Theorem 5. Let $\mu = (L, \int)$, $\tilde{\mu} = (\tilde{L}, \tilde{\int})$. We say $\tilde{\mu} \supset \mu$ if $\tilde{L} \supset L$ and $\tilde{\int} \mid L = \int$.

Theorem 5. For every abstract measure $\mu = (L, \int)$, there exists an enlarged measure $\tilde{\mu} = (\tilde{L}, \tilde{\int})$ such that $\tilde{\mu} \supset \mu$, $\tilde{\mu}$-null sets coincide with μ-null sets and $(\tilde{L})_B = L_{B'}$.

Proof. $\tilde{L}$ consists of all functions coinciding a. e. with functions of L. The remainder consists of defining $\tilde{\int}$ in the obvious manner and verifying easily the stated properties.

In view of Theorem 5 we may and shall hereafter limit our discussion to enlarged measures.

An abstract measure $\mu = (L, \int)$ is called *complete* if L is N_1-complete. We may now restate Lemma 2, for complete measures, as follows:

Lemma 3. If μ is a complete measure and $\{f_i\}_i$ is an a. e. increasing sequence of L so that $\{\int f_i\}_i$ is bounded, then $\{f_i\}_i$ converges a. e. and in N_1 to a function of L.

Proof. Since $\{f_i\}_i$ is almost everywhere increasing $\int |f_i - f_j| = \int (f_i - f_j)$ for $i > j$. Hence, since $\{\int f_i\}_i$ is bounded, $\{f_i\}_i$ is an N_1-Cauchy sequence which N_1-converges to some $f_0 \in L$, μ being complete. By Lemma 2 and Corollary 3, $f_i \to f_0$ a. e..

Theorem 6. (Lebesgue bounded convergence) If μ is complete and enlarged, f_i, $g \in L$, $f_i \to f$ a. e. and $|f_i| \leqq g$ a. e. for every i, then $f \in L$, $f_i \overset{N_1}{\to} f$ and $\int f_i \to \int f$.

Proof. By Lemma 3, if $g_i = \bigvee_{k=i}^{\infty} f_k$ a. e. and $h_i = \bigwedge_{k=i}^{\infty} f_k$ a. e. then h_i, $g_i \in L$ for every i, and $f = \overset{\infty}{\underset{i=1}{\uparrow}} h_i$ a. e.. Then $f \in L$, since $\int h_i \leqq \int g$, and μ is enlarged. For each i,

$$(14) \qquad |f-f_i| \leqq (g_i-f) \vee (f-h_i) = p_i \text{ a.e.},$$

p_i defined by (14). $0 = \overset{\infty}{\underset{i=1}{\downarrow}} p_i$ a.e. and $\lim_i N_1(f-f_i) \leqq \overset{\infty}{\underset{i=1}{\downarrow}} \int p_i = 0$, (Lemma 1). $\int f_i \to \int f$ follows from N_1-continuity of $\int$.

It follows from Theorem 6 and the definition of L_B that for μ complete and enlarged, if $g \in L$ and $f \in L_B$, then $(f \wedge g) \vee (-g) \in L$. This implies that for $f \in L_B$ and $g \in L$ with $|f| \leqq g$, we have $f \in L$. Hence $f \in L$ if and only if $|f| \in L$. Thus $L = L_1$. Also L^p is a vector space for $p \geqq 1$, because for $f_1, f_2 \in L_p$, $|f_1+f_2|^p \in L_B$ by Theorem 1, and $|f_1+f_2|^p \leqq 2^p(|f_1|^p \vee |f_2|^p) \in L$. That N_p is a seminorm will follow from the theorem involving Hölder's inequality which, together with Minkowski's inequality, can now be stated and proved within our context. However, the proofs are omitted, since they essentially follow the standard pattern.

Theorem 7 (Hölder's Inequality). Let μ be an abstract measure. If $f \in L_p$, $g \in L_q$, $(1/p)+(1/q) = 1$, $p \geqq 1$, and $fg \in L$, then

$$\int |fg| \leqq N_p(f) N_q(f) .$$

If, in addition, μ is complete and enlarged, $f \in L_p$, $g \in L_q$ imply $fg \in L$.

Theorem 8 (Minkowski's inequality). If $f, g \in L_p$, $p \geqq 1$, $f_1 = (f+g)^{p-1}f$, $g_1 = (f+g)^{p-1}g \in L$ and $f+g \in L_p$, then $N_p(f+g) \leqq N_p(f)+N_p(g)$. If, in addition, μ is complete and enlarged then $f, g \in L_p$ implies $f+g$, $f_1, g_1 \in L$.

Theorem 9. If μ is complete and enlarged, then L_p is complete for $p \geqq 1$.

Proof. Since μ is complete and enlarged, L_p is a vector space and N_p a seminorm. Let $\{f_i\}_i$ be an N_p-Cauchy sequence of L_p.

We first assume that $f_i \geqq 0$, and, without loss of generality, that $\sum_{i=1}^{\infty} N_p(f_{i+1}-f_i) < +\infty$. If $k_{nm} = \sum_{i=n}^{m} |f_{i+1}-f_i|$ a.e., then $\int (k_{nm})^p \leqq [\sum_{i=n}^{\infty} N_p(f_{i+1}-f_i)]^p$. By Lemma 3, $(k_{nm})^p$ converges (both a.e. and in N_1) to a function $(k_n)^p \in L$ as $m \to +\infty$. By a similar argument, if $f^p = \overset{\infty}{\underset{n=1}{\downarrow}} (f_n+k_n)^p$ a.e., then $f^p \in L$ and the convergence is also N_1. For $m > n$, $-k_n-k_m \leqq f_m-f_n$ and $f_n-k_n \leqq f_m+k_m$. Then $f_n-k_n \leqq f \leqq f_n+k_n$ a.e.. Consequently $N_p(f-f_n) \leqq N_p(k_n) \leqq \sum_{i=n}^{\infty} N_p(f_{i+1}-f_i) \to 0$ as $n \to +\infty$.

For an arbitrary N_p-Cauchy sequence $\{f_i\}_i$, $N_p(f_i^+-f_j^+) \leqq N_p(f_i-f_j)$ and $N_p(f_i^--f_j^-) \leqq N_p(f_i-f_j)$. Hence $\{f_i^+\}_i$ and $\{f_i^-\}_i$ converge (both N_p and a.e.)

to functions $f^{+} \geqq 0$ and $f^{-} \leqq 0$ respectively and $N_p(f^{+}-f^{-1}-f_i) \leqq N_p(f^{+}-f_i^{+})+N_p(f_i^{-}-f^{-}) \to 0$.

4. *Completion of a measure.* Let F be any real vector space and π a seminorm of F. It follows from the general theory of topological vector spaces and of uniform spaces ([7, p. 12] and [5, pp. 150-155]) that F can be completed with respect to π into a vector space $\tilde{F}$ containing F with a seminorm $\tilde{\pi}$ which extends π and F is $\tilde{\pi}$-dense in $\tilde{F}$. $\tilde{F}$ is obtained in a manner analogous to the way one obtains the reals by the Cantor process as Cauchy sequences of rational numbers. $\tilde{F}$ consists of all π-Cauchy sequences of F and linear combinations in $\tilde{F}$ are ordinary linear combinations of sequences. $\tilde{\pi}(\{f_i\}_i) = \lim_i \pi(f_i)$ for a π-Cauchy sequence $\{f_i\}_i$. Any continuous real linear functional T on F can be extended uniquely as $\tilde{T}(\{f_i\}_i) = \lim_i T(f_i)$ to $\tilde{F}$. If $f \in F$, the Cauchy sequence $\tilde{f} = \{f, f, f, \cdots\}$ corresponds to f under the imbedding of F in $\tilde{F}$. It should be noted that the method just described is not exactly analogous to the Cantor process. In the Cantor process Cauchy sequences which converge to "the same element" are identified, while here each Cauchy sequence is considered an individual element. This results in anomalies such as the same sequence converging to many elements, with which we have already come to terms. Connected with this is also the fact that the completion of a non-Hausdorff seminormed vector space is not unique in any reasonable sense, while that of a metric space is.

If $\mu = (L, \int)$ is an abstract measure, $\tilde{L}$, $\tilde{\int}$, $\tilde{N}_1$ the corresponding extended objects under the above completion, we may define $|\{f_i\}_i| = \{|f_i|\}_i$ and have $\tilde{\int}$ positive in the sense that $\tilde{\int}|\{f_i\}_i| \geqq \int\{f_i\}_i$. In the same manner $\tilde{N}_1(\{f_i\}_i) = \tilde{\int}|\{f_i\}_i|$.

Theorem 10. For every enlarged abstract measure $\mu = (L, \int)$ there exists a complete enlarged measure $\bar{\mu} = (\bar{L}, \bar{\int}) \supset \mu$ so that $\bar{N}_1 = \bar{\int}|f|$ extends N_1, L is $\bar{N}_1$-dense in $\bar{L}$, $(\bar{L})_B = L_B$ and $\bar{\mu}$-null sets coincide with μ-null sets.

Proof. We divide this proof into four important stages, which we label individually.

A. Construction of $\bar{\mu}$. Let $\bar{L}$ be the set of all functions on S that are a. e. limits (in the sense of the remarks preceding Corollary 4) of N_1-Cauchy sequences of L. These sequences are nothing but elements of $\tilde{L}$ (see remarks preceding this theorem). Clearly $\bar{L}$ is a vector lattice. Also, that $\bar{L}$ satisfies (1) follows from $|f \wedge 1 - g \wedge 1| \leqq |f-g|$. If $\{f_i\}_i \in \tilde{L}$ and $f = \lim_i f_i$ a. e. then we define $\bar{\int} f = \tilde{\int}\{f_i\}_i$ and $\bar{N}_1(f) = \bar{\int}|f| = \tilde{N}_1(\{f_i\}_i)$. $\bar{\int}$ and $\bar{N}_1$ are well defined, for if another $\{g_i\}_i \in \tilde{L}$ converges a. e. to f, then $N_1(f_i - g_i) \overset{i}{\to} 0$, by Corollary 4. We define $\bar{\mu}$ as $(\bar{L}, \bar{\int})$. Several properties of $\bar{\int}$ are readily verified. It is positive. If $f, g \in \bar{L}$ and $f \geqq g$ a. e. (in the sense of μ) then $\bar{\int} f \geqq \bar{\int} g$, for $f \geqq f \wedge g$ everywhere and $0 \overset{\text{a.e.}}{=} (f \wedge g) - g \in L$, implying $\bar{\int} f \geqq \bar{\int} f \wedge g = \bar{\int} g$. In particular, if $f = g$ a. e., $\bar{\int} f = \bar{\int} g$.

The construction of $\bar{L}$ out of $\tilde{L}$ does not provide a convenient single valued mapping of one of these spaces onto the other, for each element of $\tilde{L}$ converges a. e. to many functions, and every $f \in \bar{L}$ is the limit a. e. of many elements of $\tilde{L}$. However, the value of our ability to assign to each $f \in \bar{L}$ some element of $\tilde{L}$ which converges a. e. to f has already been demonstrated.

B. $\bar{L}$ is $\bar{N}_1$-complete and clearly enlarged, and L dense in $\bar{L}$. For completeness, let $\{f_i\}_i$ be an $\bar{N}_1$-Cauchy sequence and each f_i an a. e. limit of $\{g_i^j\}_j \in \tilde{L}$. Then $\{\{g_i^j\}_j\}_i$ is an $\tilde{N}_1$-Cauchy sequence of $\tilde{L}$, which by completeness of $\tilde{L}$ converges to some $\{k_j\}_j \in \tilde{L}$. Let $f = \lim_j k_j$ a. e.. Then $\bar{N}_1(f - f_i) = \tilde{N}_1(\{k_j\}_j - \{g_i^j\}_j) \xrightarrow{i} 0$. L is dense in $\bar{L}$ for if f is the limit a. e. of $\{f_i\}_i \in \tilde{L}$, $\bar{N}_1(f - f_j) = \tilde{N}_1(\{f_i\}_i - \tilde{f}_j) \xrightarrow{j} 0$.

C. $\bar{\mu}$ satisfies (4). Let $0 = \overset{\infty}{\underset{i=1}{\downarrow}} k_i$, $k_i \in \bar{L}$. Then $\{k_i\}_i$ is an $\bar{N}_1$-Cauchy sequence, being monotone with bounded integrals. We show $0 = \overset{\infty}{\underset{i=1}{\downarrow}} \bar{\int} k_i$.

(i) We first assume that each $k_i = \overset{\infty}{\underset{j=1}{\uparrow}} f_i^j$ a. e., $0 \leqq f_i^j \in L$. $\bar{N}_1(k_i - f_i^j) \xrightarrow{j} 0$ for each i. For every i choose $f_i^{j_i}$ so that $\bar{N}_1(k_i - f_i^{j_i}) < 1/2^i$. $\{f_i^{j_i}\}_i$ is an N_1-Cauchy sequence, since $\{k_i\}_i$ is an $\bar{N}_1$-Cauchy sequence, and $f_i^{j_i} \to 0$ a. e. since $f_i^{j_i} - k_i$ a. e.. By Theorem 2, $\{f_i^{j_i}\}_i$ N_1-converges to 0 and $\{k_i\}_i$ $\bar{N}_1$-converges to 0.

(ii) In the general case $k_i = \lim_j f_i^j$ a. e. $\{f_i^j\}_j$ L, $f_i^j \geqq 0$. We may assume $\sum_{j=1}^{\infty} N_1(f_i^{j+1} - f_i^j) < +\infty$. For every two positive integers m and n, $m > n$ and each i, let

$$h_i^{mn} = f_i^n + \sum_{j=n}^{m-1} |f_i^{j+1} - f_i^j| \tag{15}$$

$\{h_i^{mn}\}_m$ is an increasing $\bar{N}_1$-Cauchy sequence which converges a. e. to a function $h_i^n \in \bar{L}$, and $k_i = \overset{\infty}{\underset{n=1}{\downarrow}} h_i^n$ a. e.. Now, if $g_r = \bigwedge_{i \leqq r,\, n \leqq r} h_i^n$, $g_r \geqq k_r$ a. e..

What is more $0 = \overset{\infty}{\underset{r=1}{\downarrow}} g_r$ a. e.. If we show $0 = \overset{\infty}{\underset{r=1}{\downarrow}} \overline{\int} g_r$, then $\overline{\int} k_r - \overline{\int} g_r \to 0$. But, by (15) each h_i^n is of the type that each k_i is in (i). Hence, so is g_r. $0 = \overset{\infty}{\underset{r=1}{\downarrow}} \overline{\int} g_r$ follows then from (i).

D. $L_B = (\overline{L})_B$ and $\overline{\mu}$ yields the same null sets as μ. Since μ is enlarged $L_B = L_{B'}$, hence $L \subset \overline{L} \subset L_B$ and $(\overline{L})_B = L_B$.

A μ-null set is clearly $\overline{\mu}$-null. Let B be a $\overline{\mu}$-null set. For each $\epsilon > 0$ there is a sequence $\{k_i\}_i$, $k_i \in \overline{L}$ so that $\chi_B \leqq \overset{\infty}{\underset{i=1}{\uparrow}} k_i$ and $\overline{\int} k_i < \frac{\epsilon}{2}$. $\{k_i\}_i$ is an $\overline{N}_1$-Cauchy sequence and, since $\overline{\mu}$ is complete, it $\overline{N}_1$-converges to a function $k \in \overline{L}$. By Corollary 4 $k_i \overset{\text{a. e.}}{\to} k$. $\overline{\int} k \leqq \frac{\epsilon}{2}$, $k = \lim_i f_i$ a. e. for some $\{f_i\}_i \in \tilde{L}$, and we may assume $\sum_{i=1}^{\infty} N_1(f_{i+1} - f_i) < \infty$. Then, if $g_i = f_i + \sum_{j=i}^{\infty} |f_{j+1} - f_j|$ a. e., $g_i \in \overline{L}$, $k = \overset{\infty}{\underset{i=1}{\downarrow}} g_i$ a. e. and $g_i \overset{N_1}{\to} k$, (Corollary 4). Then there exists an i so that $\overline{\int} g_i < \epsilon$. But

$$\chi_B \leqq k \leqq g_i = \overset{\infty}{\underset{n=i}{\uparrow}} (f_i + \sum_{j=i}^{n} |f_{j+1} - f_j|) \quad \text{a. e.,}$$

and by redefining each f_i on a μ-null set we have

$$\chi_B \leqq \overset{\infty}{\underset{n=i}{\uparrow}} (f_i + \sum_{j=1}^{n} |f_{j+1} - f_j|)$$

everywhere. Hence B is μ-null. This completes the proof of Theorem 10.

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Minneapolis, Minnesota

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

THE NEW WORLD OF MATHEMATICS

James H. Zant

A revolution is underway in the field of mathematics and mathematics teaching. This involves the amount of mathematics known and used in the modern world and it also affects the content and point of view from which mathematics is, or should be, taught at the elementary level, at the secondary level and at the college level. There is a new world of mathematics and the revolution has been going in the content of the field with ever increasing rapidity so that now the new mathematics forms the bulk of the total accumulation.

Though mathematics has been developing for 5,000 years the subject has never been as lively as it is today. The pace of mathematical discovery and invention has accelerated amazingly during the last twenty years. It has been said that mathematics is the only branch of learning in which all of the theories of two thousand years ago are still valid. Hence there is little that is basically wrong with the mathematics of the past and the vitality and vigor of present day mathematical research indicates that the sheer bulk of mathematical developments is staggering.

While mathematics is basically a logical structure which has no need for applications to justify its existence, applications are well known and numerous. This is true of the most abstract parts of the subject as well as many sections which may seem, at the time of their creation, to be mere play-things invented purely for the mathematician's enjoyment. For example, the decimal number system with the base ten using the ten digits 0, 1, 2, 3, $\cdots$, 9 has been used by nearly all people all over the world. Humans were born with ten fingers and almost universally they have counted to ten and started over. But number systems to other bases have been invented by mathematicians. One of these, to the base two, is called the binary number system and may have been invented for the pleasure of the mathematicians. This system uses only two digits, 0 and 1. Numbers are then 1, 10, 11, 100, 101 instead of 1, 2, 3, 4, 5. This number system is not practical for counting things in ordinary life or for writing numbers.

However, it is not a fad. It has a new importance and has made a terrific impact on the modern world because this method of counting is of necessity that used by the powerful modern digital computors. The reason is that they count on electrical relays, which have only two "*fingers*", "off" and "on".

We find mathematics being applied to all sorts of situations; in the social sciences; to the distribution of manufactured products; to traffic flow; bacterial growth is expressed as an exponential function; the same function is used to determine the half-life of radio-active substances. It is now thought that the area of topology can be used in describing the surface of the living cell. A new kind of algebra to represent the thinking process is being sought by neurophysiologists. This process is by no means random, but is not entirely methodical. Instances of applications of mathematics and sought for applications could be continued indefinitely..

From all of this it is clear that the world of today demands more mathematical knowledge than in the past. No one can predict with accuracy his future profession or how much mathematics will be demanded in a particular profession 25 years from now. It is important to learn mathematics with understanding so that students of today will be able in later life to learn the new mathematical skills which the future will surely demand of many of them.

A language barrier exists here, however. The language for the communication of mathematical ideas is largely in terms of symbols and words which the public cannot understand and therefore does not listen. This is not true in the other sciences as was illustrated in 1956 when two theoretical physicists, Chen Ning Yang and Tsung Dao Lee demolished the principle of parity. The press did a commendable job of describing the abstruse physical ideas and the implications involved, but said nothing about the mathematical theory of groups on which the conclusions were based. Physicists, with tongue in the cheek reservations, talk about atoms and subatomic particles as if they were little marbles, but there is no popular terminology for talking *about* mathematics.

However, some mathematicians have always been able to communicate – often better to children than to adults. The late Dr. Edward Kasner of Columbia University regularly lectured to kindergarten classes on infinite sets. The children easily reconciled themselves to the notion of infinity and got the fundamental ideas of set theory faster than some of his undergraduate students. Children seem naturally attuned to mathematical abstractions, perhaps because it is not unlike pure fantasy. One of the best-loved of all children's books is *Alice's Adventures in Wonderland,* first conceived as stories told to children by a professional mathematician, Charles Lutwidge Dodgson who used the pen name, Lewis Carroll. Presently one of our outstanding young mathematicians, Dr. Paul C. Rosenbloom of the University of Minnesota, is successfully introducing mathematical concepts, particularly in the field of number theory, to grade school children.

The Great Teacher said "Except ye become as a little child, ye shall in no wise enter the Kingdom of Heaven." Perhaps this is the key to teaching real and exciting mathematics to the young students in our schools.

The New Mathematics in the Schools

By some curious coincidence the field of mathematics, which has expanded continuously over a period of 5,000 years, became a static, almost stagnant, subject in the classrooms of both the high schools and colleges of this country. Subject matter for courses in algebra, geometry, trigonometry, analytical geometry and the calculus extending from grade 9 through undergraduate college curriculum was crystallized into its present form approximately 60 years ago and has changed little since that time. The subject matter, which was developed for the most part over 300 years ago (in the case of plane and solid geometry it was over 2,000 years ago), was chosen and its presentation organized in accordance with an attitude toward mathematics which no longer holds and which nas been discarded by present day mathematicians. With this curriculum and point of view mathematics is presented mainly as a collection of slightly related techniques and manipulations. The profound, yet simple concepts get little attention. It has been said that "if art appreciation were taught in the same way, it would consist mostly of learning how to chip stone and mix paint."

The type of mathematics and mathematics courses referred to above are all too common in high school and college today. We shall refer to this as *Traditional Mathematics*. This will be in contrast to the terms *Modern Mathematics* or *Contemporary Mathematics* now coming into common use for courses and curricula which include new concepts and which are taught from a different point of view.

Thus the whole field of mathematics, traditional and modern, is very large and all of it basically good mathematics. We cannot teach all of it in high school and college. It is also almost impossible to separate this subject matter into high school and college mathematics at this stage in the development of mathematical curricula; there will always be an overlap. We hope that by the use of both new concepts and conscious emphasis on the point of view that mathematics is concerned with abstract patterns of thought or a mathematical structure that students will acquire an understanding of and an interest in mathematics as a cultural as well as an applied subject.

It should be noted at once that there is no sharp dividing line between "traditional" mathematics and "modern" mathematics. The development of mathematics has been continuous; the newer ideas have grown out of the older; and many teachers have been presenting traditional content from a point of view accurately described as modern.

The approach of considering the concepts of mathematics from many points of view is one of the characteristics of modern mathematics that has made it a powerful tool for understanding mathematics and also in applying it to different situations. We can solve problems in new ways, solve some problems that we have been unable to solve before and discover properties that have eluded us before.

The case for modern mathematics rests on a number of points. Mathematics is a dynamic, growing field and has far outrun the curriculum both at the school and college level. The traditional curriculum has not resulted in our students learning enough mathematics or good mathematics and does not emphasize the fact that the developments and applications of mathematics are not only important but indispensable to human progress. The use of newer concepts not only increases the power of the students in applying mathematics, but also enhances his understanding of the subject. The necessity for a thoroughly reorganized program oriented to the needs of the second half of the twentieth century is dictated by the immediate and urgent needs for mathematicians as well as scientists and engineers who understand basic modern mathematics and know how to use it to solve the problems incident to a wide variety of the aspects of modern civilization.

Trends in Mathematics and Mathematics Teaching

Mathematics is a basic element in the whole program of revised science education. The crisis in science education is a real one. It is not an invention of the newspapers or the scientists or the Pentagon. While the U.S.S.R. is not the "cause" of the crisis, it has served as a rude stimulus to awaken us to reality. The cause of the crisis is our breath-taking movement into a new technological era. This movement into a new phase of man's long struggle to control his environment involving the use of nuclear energy, exploration of outer space, revolutionary creations and applications of both mathematics and science can test to the utmost our adaptive capacities. Education, at all levels, must meet this challenge. We need a well trained citizenry, but we need, perhaps more, an ample supply of high caliber scientists, mathematicians and engineers.

To meet this basic need two trends in mathematics curricula and teaching are significant. This represents the revolution in mathematics teaching. The trends are acceleration and modernization of content and point of view.

Acceleration

Generally acceleration is taking place in the secondary schools independently of any change in the curriculum. Perhaps the most common sort of acceleration now underway consists of allowing or requiring students to start algebra in the eighth grade and then proceed with other traditional mathematics courses in order. Administratively this is the easiest way to do it and it is becoming almost a national trend. Other methods of acceleration are to encourage students to take courses in mathematics during the summer and to separate the class according to ability and allow the better students to preceed through the courses at a faster rate.

All of these and other methods are in use and have some merit. There are also certain disadvantages. One of these is that secondary mathematics is organized as a four year sequence at most and acceleration allows the student to complete this sequence by the end of the junior year. This makes it necessary to offer a course in analytics and calculus during the senior year or provide no mathematics for the students during the last year in high school. Few committees or groups of professional mathematicians recommend that calculus be taught in high school, but it is being done in a great many schools. It is almost a national trend. The fact is that there is no other available course or textbook with which teachers are familiar.

A second disadvantage of accelerating students is that it is usually assumed that the present curriculum in high school mathematics is a good one. This is contrary, of course, to the contention of this discussion. I feel that secondary mathematics from the first grade up, but especially from grade 7 through 12, should be reorganized to include a number of modern concepts and that it should be taught from the point of view that mathematics is concerned with abstract patterns of thought. If the mathematics curriculum is so organized and taught, then it will be possible to have enough material in all grades through the twelfth to give the students a broad understanding of the true meaning of the subjects and to furnish him an understanding of the processes and skills necessary to live and work in the modern world.

Modernization of the Mathematics Curriculum

Certain characteristic features of a modern mathematics program may be stated briefly as follows: Courses are designed for college capable students, but may be used with less talented students if they are given more time. New concepts and a different point of view are used because such knowledge is needed by the students. Modern concepts lead to a clarification of the subject. Changes will not be radical enough to cause a great disturbance with teachers or students. While changes will help students meet present needs, they also provide an understanding of mathematics for future change and development.

While it is not possible to give a complete outline of modern mathematics for the five grades under discussion here, such outlines have been made and one set of textbooks in preliminary form is being used on an experimental basis this year in 22 Oklahoma schools. In this series of books grades 7 and 8 deals with the structure of arithmetic and the real number system as a progressing development and with metric and non-metric relations in geometry. Though these ideas are associated with their applications, materials involves experience with and appreciation of *abstract concepts,* the role of *definition,* the development of *precise vocabulary* and *thought, experimentation* and *proof.* Notable for it's absence is the area of socialized arithmetic dealing with paying bills, insurance, taxes and the like. These latter situations are not real to students at this age level, but they are capable of learning the more fundamental concepts of number and measurement.

The ninth grade is basically algebra. Students will explore the behavior of numbers and invent new numbers to describe new situations. They will find that all the manipulations with symbols can be made understandable and that they "hang together" in a very satisfactory way. The mathematics is sound; it will not be necessary to unlearn parts of it later. It is this emphasis on a clear cut and mathematically sound picture of the structure of algebra which distinguishes this course from the more traditional ones.

New concepts introduced are simple, but extremely useful in getting a better understanding of algebra. They include sets and operations on sets of numbers, phrases and sentences (for expressions and equations), the properties (commutative, associative and distributive) of numbers, the real number line, order and the coordinate (rectangular) system.

In like manner, the tenth grade is concerned entirely with geometry, but designed in such a way that the algebra studied in the ninth grade becomes a more useful tool. The course includes a few chapters on solid geometry (no separate course in solid geometry is recommended) and a single chapter on analytic geometry. The conviction is that the traditional content of Euclidean geometry deserves the prominent place it has always held in the high school. Changes have been made only when the need for them appears compelling. The postulates used (they are those of G. D. Birkhoff rather than the more sophisticated system devised by Hilbert) assumes a knowledge of the number system treated adequately in the ninth grade. After the first few chapters, which teachers must learn the first time with the students, it is not anticipated that there will be a great difference from the traditional course. It will, however, be more precise and more logically presented.

The eleventh grade course will consist of topics which usually appear in intermediate algebra and trigonometry, but it is not a course in "mathematics made easy" wherein the students learn how to *do* certain things. Since this is usually an elective course in high school, some selection of the students can be assumed. Hence inherent difficulties in understanding are candidly appraised and forthrightly explained in terms appropriate for students at this level. A controlling consideration is the desire to advance the students understanding of the number system. Mathematical proofs are used, but it is not necessary that they be unduly rigorous.

An appeal is often made to the student's intuition and he is led by an inductive approach to make and test conjectures about the nature of the principles to be proved. New symbolism is used only when it serves to convey meaning more accurately and succinctly than could be done by other means.

The material for the twelfth grade, traditionally solid geometry and college algebra or college algebra and trigonometry, departs more radically from the traditional than for the other courses. While some of the so-called modern concepts are used, the subject matter itself can probably be more accurately described as "traditional mathematics which is developed from a more up-to-date point of view."

The title of this course is usually *Elementary Functions* which may be thought of as a more sophisticated term for *Algebra*. It deals with polynomial functions, exponential and logarithmic functions, trigonometric functions, algebra of matrices, and perhaps an introduction to abstract algebra.

Alternative subject matter has been and is being written. There is a book on *Introductory Probability and Statistical Inference* which is suggested for the second semester of the twelfth grade. Generally the recommendations as to the content of the twelfth grade course are much more flexible than for the other grades. However, as has been said, none of the recommendations include a course in calculus.

Summary and Implications

1. There is a new world of mathematics. This affects us in our daily lives, not only for people who use mathematics in the many ramifications of our scientific age, but also in what we should be teaching in our schools to prepare present day students to live and work in the world of the future.

2. The mathematics curriculum of the schools must be revised to meet this challenge.

3. This revision process is already well underway so that schools in Oklahoma are now teaching courses involving the new mathematics.

4. This revision is not drastic in terms of courses and topics included. However, it is fundamental in terms of the use of new concepts and point of view in teaching and learning mathematics.

5. All possible help must be provided both for teachers now in service and for those who are preparing to teach this new and exciting material.

The future in the field of mathematics and through it all of the new developments of our modern scientific age, was never brighter. Fundamental research is progressing at an amazing rate. The center of much of this activity is shifting to the United States. In like manner the reorganization of the curriculum in mathematics is being pursued with vigor and enthusiasm and from a most realistic point of view. This reorganization is being supported and contributed to by many of our finest mathematicians. They are working cooperatively and in equal numbers with outstanding teachers of mathematics in the schools. Administrators realize the importance and significance of this effort and are supporting it with enthusiasm. Finally, perhaps the most important characteristic of the whole movement is the direct involvement of teachers in the classroom at local levels. Those of us who are familiar with what appeared to be promising efforts over the last 50 years realize that success is often less than anticipated. Perhaps the closer involvement of administrators and teachers in a cooperative effort at local levels may be the difference which will insure the success of this program.

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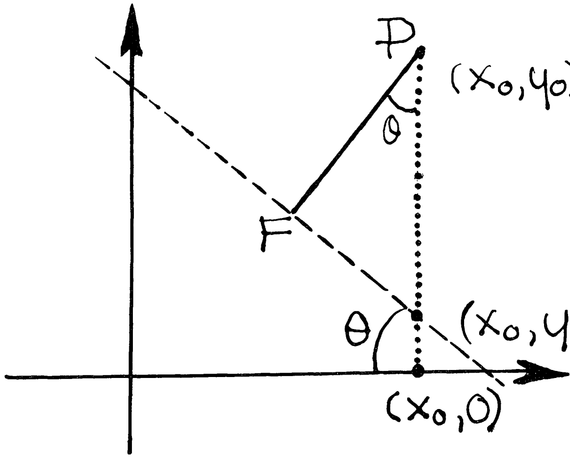
POINT TO LINE DISTANCE

William R. Ransom

The derivation of the formula for the distance from a point to a line usually employs a complicated diagram: a simpler diagram will serve.



Let (x_0, y_0) be the point and $y = mx + b$ be the line. Then $PF = (y_0 - y_1)\cos\theta$. Since $\tan\theta = m$, $\sec\theta = \sqrt{1+m^2}$, and $\cos\theta = 1/\sqrt{1+m^2}$. Since $y_1 = mx_0 + b$, $PF = (y_0 - mx_0 - b)/\sqrt{1+m^2}$.

An even simpler diagram, which omits the dotted line, will serve. If the point (x_0, y_0) is not on the line we have

$$Ax + By + C = 0,$$
$$Ax_0 + By_0 + C = L \neq 0.$$

Subtraction gives

$$A(x - x_0) + B(y - y_0) = 0$$

while

$$B(x - x_0) - A(y - y_0) = 0$$

represents the line on which lies the perpendicular distance PF.

Solving this last pair of equations for the intersection, F, and using the abbreviation $H^2 = A^2 + B^2$, we get

$$x - x_0 = -AL/H^2 \quad \text{and} \quad y - y_0 = -BL/H^2$$

and as these are the rise and run of the distance PF, we get

$$(PF)^2 = (A^2L^2 + B^2L^2)/H^4$$

whose square root gives the distance as L/H, which is

$$(Ax + By + C) \,/\, \sqrt{A^2 + B^2}\,.$$

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MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

THE EXISTENCE OF INTEGERS LESS THAN p BELONGING TO ep^{r-1} (mod p^r)

John and Margaret Maxfield

INTRODUCTION: Take p to be an odd prime, and r an integer ≥ 1. In [1] Lebesgue stated that if a is a primitive root (mod p) then either a or $a' \equiv a^{p-2}$ (mod p) ($a' < p$) is a primitive root (mod p^r). In this note we prove that if a belongs to $e > 1$ (mod p), then a or $a' \equiv a^{e-1}$ (mod p) with $a' < p$ belongs to $p^{r-1}e$ (mod p^r).

Definition 1. An integer a *belongs* to the exponent e (mod m) if $a^e \equiv 1$ (mod m) and if e divides every x such that $a^x \equiv 1$ (mod m).

Definition 2. An integer a is a *primitive root* (mod p^r) if it belongs to $(p-1)p^{r-1}$ (mod p^r).

Lemma 1. If a belongs to e (mod p^r), then it belongs either to e or to pe (mod p^{r+1}).

Proof: We have for some integer k

$$a^e = 1 + kp^r .$$

Then $a^{pe} = (1+kp^r)^p = 1 + pkp^r + \cdots \equiv 1$ (mod p^{r+1}), so that if x is the exponent to which a belongs (mod p^{r+1}), x must divide pe. But also $a^x \equiv 1$ (mod p^{r+1}) implies $a^x \equiv 1$ (mod p^r), whence e divides x. Then x equals e or pe.

Lemma 2. If a belongs to e (mod p^r) and to pe (mod p^{r+1}), then a belongs to p^2e (mod p^{r+2}).

Proof: We have $a^e = 1 + kp^r$ with $p \nmid k$, so that $a^{pe} \equiv (1+kp^r)^p \equiv 1 + kp^{r+1} \not\equiv 1$ (mod p^{r+1}).

THEOREM: Let $a < p$ belong to $e > 1$ modulo an odd prime p. Then either a or a', where $a' < p$ and $a' \equiv a^{e-1}$ (mod p) belongs to ep^{r-1} (mod p^r).

Proof: Suppose $a^e \equiv 1$ (mod p^2). We have for some integer c

$$a' = a^{e-1} + cp ,$$

so that $(a')^e = (a^{e-1}+cp)^e \equiv (a^e)^{e-1} + ea^{e-1}cp \equiv 1 + ea^{e-1}cp \pmod{p^2}$. If $(a')^e \equiv 1 \pmod{p^2}$, then, it follows that $c \equiv 0 \pmod{p}$. In that case, $aa' = a^e + acp \equiv 1 \pmod{p^2}$, but since $e > 1$, $a > 1$ and both a and a' are $< p$, so $aa' < p^2$, giving us a contradiction.

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CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

COMMENT ON A NEW LOOK AT AN OLD PROBLEM

George M. Bergman

Mr. Wingo's attack on integral solutions of the equations:

$$x^2 + y \ \ = a$$

$$x \ \ + y^2 = b$$

in the May-June 1959 issue of this magazine was pleasantly ingenious, but there was one error in logic: he shows, near the middle of p. 276, that $x-y$ will be slightly larger than $\sqrt{a}-\sqrt{b}$, and concludes that it will be the factor of $a-b$ just greater than this. Actually, there may be a smaller factor of $a-b$ that, though not the correct value of $x-y$, is larger than $\sqrt{a}-\sqrt{b}$. as in this case:

$$x = 13\ , \quad y = 4\ , \quad a = 173\ , \quad b = 29\ ,$$

$$a-b = 144 \text{ Factors: } 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 16,\ 18,\ 24,\ 36,\ 48,\ 72$$

$$\sqrt{a}-\sqrt{b} = 13.15 - 5.38 = 7.77 \text{ (approx.)} \quad x-y \overset{?}{=} 8 \text{ No!}$$

The factorization $144 = 8 \times 18$ could have been eliminated beforehand, however, by the fact that it would make both $x-y$ and $x+y-1$ even, which is impossible for integral x and y. The only factorizations we need have considered were 16×9, 48×3, and 144×1.

Surprizingly, it can be proven that this criterion will eliminate all "false" cases.

The proof is as follows:

Let $x-y = Q$, but let there be some smaller factor of $a-b$, $Q-p$, which we fear might be mistaken for it (as 8 was above).

As Q and $Q-p$ can have no factors but factors of p, and $a-b$ is a multiple of each, we can write $a-b = r[Q(Q-p)/p]$ with an integral r. Thus

$x+y-1=(a-b)/Q=r[(Q-p)/p]$. Since $x+y-1>x-y$, $r[(Q-p)/p]>Q$. This necessitates $r>p$, and we may write $r=p+s$ where s is a positive integer.

Now let us assume that $Q-p \geqq \sqrt{a}-\sqrt{b}$.* Since $x<\sqrt{a}$, $Q-p>x-\sqrt{b}$, and $\sqrt{b}>x+p-Q$. Substituting $Q=x-y$, we get $\sqrt{b}>p+y$, and $b>p^2+2py+y^2$. Subtracting $b=x+y^2$, we have $0>p^2+2py-x$.

Let us now express x and y in terms of p, Q, and s, for as these relate the "wrong" factorization of $a-b$ to the "right" one, it is they that will be significant when we use the parity criterion. We know that $x-y=Q$. Also, $x+y-1=(p+s)[(Q-p)/p]=Q-p+(sQ/p)-s$. Solving,

$$y=\frac{sQ}{2p}-\frac{p}{2}-\frac{s}{2}+\frac{1}{2} \quad \text{and} \quad x=Q+\frac{sQ}{2p}-\frac{p}{2}-\frac{s}{2}+\frac{1}{2}.$$

We substitute:

$$0>p^2+sQ-p^2-sp+p-Q-\frac{sQ}{2p}+\frac{p}{2}+\frac{s}{2}-\frac{1}{2}$$

$$0>sQ-sp-\frac{sQ}{2p}+\frac{s}{2}-Q+p+\frac{p}{2}-\frac{1}{2}$$

$$0>[(1-\frac{1}{2p})s-1](Q-p)+\frac{p-1}{2}$$

As $Q>p\geqq 1$, this cannot be satisfied by any $s\geqq 2$. (Examine the signs of the various terms in parentheses, if this is still not clear the reader may demonstrate it by subtracting $0\leqq(p-1)/2$, dividing the inequality by $Q-p$, which is positive, transposing the -1, and dividing by $(1/2)\leqq[1-1/2p]$). We shall now show that our criterion will eliminate all cases with $s=1$.

In such a case the "correct" factorization of $a-b$ is $Q\cdot(p+1)[(Q-p)/p]$ (using $p+1$ for r). For the second factor to be integral, Q must be divisible by p; we can write it pq. $a-b$ is now $pq\cdot(p+1)(q-1)$. For the factors to have opposite parity, p and q must have the same parity. But then in the "wrong" factorization, $a-b=(Q-p)\cdot(p+1)(Q/p)=p(q-1)\cdot q(p+1)$, the factors will both be even (like 8 and 18), and our criterion will eliminate it.

It should be noted that Mr. Wingo has simply replaced the trial-and-error method of substituting various values for x and y with the trial-and-error method of factoring $a-b$. Practically, of course, this completely solves the problem, as factorization is far easier, and there are tables of results; but from an abstract point of view there is no gain in explicitness. This is typical of mathematics where we may replace an endless series with an expression in roots, sines, exponentials,... substituting for one infinite expression others that have been defined, studied, and tabulated.

*i. e. Mr. Wingo's criterion will not eliminate $Q-p$ as the "correct" factor.

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COMMENT ON A NEW LOOK AT AN OLD PROBLEM BY CHARLES E. WINGO JR.

Jeff Cheeger

We are given $x^2 + y = a$, $y^2 + x = b$ and $a > b$ which implies $x > y$. x and y are integers. It will be shown that x must always be the greatest integer contained in the square root of a. Let $[\sqrt{a}]$ be $(m+1)$. Assume $x \leqq m$. We have $y = a - x^2 \geqq (m+1)^2 - x^2 \geqq 2m + 1 > m \geqq x$ which contradicts $x > y$. Hence $x = [\sqrt{a}]$. If we are given that the solutions exist then the problem is solved. Example: $x^2 + y = 6472$, $x = [\sqrt{6472}] = 80$, $y = 6472 - 6400 = 72$. If we are not given that the solutions exist then we must check in the second equation, and of course if $a - x^2 > x$ no solutions exist. Example: $x^2 + y = 6492$, $x = 80$ as before but $y = 92$ which is greater than x so no solutions exist.

This method can easily be generalized to solve for integral values the equation $x_n^{C_n} + x_{n-1}^{C_{n-1}} + x_{n-2}^{C_{n-2}} + \cdots + x_{n-m}^{C_{n-m}} = k$ where $n > m$, $x_p \geqq x_{p-1}$ and $C_p > C_{p-1}$ for all $p \leqq n$. To show that if $(m+1)$ is $[\sqrt[C_n]{k}]$ then $x_n = m+1$, assume $x_n \leqq m$. We have

$$x_{n-1}^{C_{n-1}} + x_{n-2}^{C_{n-2}} + \cdots + x_{n-m}^{C_{n-m}} = k - x_n^{C_n} \geqq (m+1)^{C_n} \geqq$$
$$C_n m^{C_n - 1} + \frac{C_n(C_n - 1)}{2!} m^{C_n - 2} + \cdots + 1$$

which readily implies $x_{n-1} > m \geqq x_n$; again a contradiction. Therefore $x_n = [\sqrt[C_n]{k}]$, $x_{n-1} = [\sqrt[(C_{n-1})]{k - x_n^{C_n}}]$ and so on. Example: $x^5 + y^3 + z^2 = 7910$, $x = [\sqrt[5]{7910}] = 6$, $y = [\sqrt[3]{7910 - 7776}] = 5$, $z = [\sqrt[2]{134 - 125}] = 3$. These values are substituted in the other equations if any to see if they are also satisfied.

Erasmus Hall H. S.
Brooklyn, N. Y.

Dimensions, Units, and Numbers. By Renee G. Ford and Ralph E. Cullman. Bureau of Publications, Teachers College, Columbia University, 1959. ix + 49 pp. \$1.00.

This monograph is one of a series that is being sponsored by the Science Manpower Project of Teachers College, Columbia University, under the direction of Frederick L. Fitzpatrick, Head of the Department of Teaching of Science. Its place in the mathematical literature is more clearly indicated by its complete title, *Dimensions, Units, and Numbers in the Teaching of Physical Sciences.*

The monograph is intended as a resource for teachers of the physical sciences and attempts to assist them in developing the dimensional and unit analysis approach to quantitative concepts and problems in these sciences. This objective is clearly stated in the editor's preface. Although the monograph is intended for secondary school teachers in the sciences, it likely will be of value also in colleges. In this connection it would be helpful to have a similar volume addressed to the college student, especially the first year engineering student.

The content is not new and there seem to be no novel ways of doing things developed in the book. However, even though the content is standard, it is valuable to have these aspects of dimensional analysis brought together in convenient form and emphasized. Most high school science teachers and most beginning science and engineering students in colleges and universities could benefit from reading this monograph and keeping it for handy reference.

From a technical point of view there are a few items that are worth mentioning. I feel that in a book of this sort there should be some discussion of the concept and nature of the radian. This has been omitted. There is also no mention of the difference between the pound-foot (cm-dyne) (torque) and the foot-pound (dyne-cm) (work, energy). In discussing the gas laws on p. 44 the authors refer to the quantity of the gas. Does this mean the mass, weight, or volume of the gas? If one of these is intended, the discussion would be clarified if the proper meaning were identified.

A few technical and typographical changes would improve the monograph. On page 25 a parenthesis is misplaced. Because of the nature of the book a change should be made on page 26, line 6. It should read, 300 dynes = 300 dynes $\times$ 1 poundal/(1.39 $\times$ 10^4 dynes) = $\cdots$

What is said at the bottom of page 33 about powers of 10 is true. But the authors undoubtedly meant *integral* powers of ten. In discussing the rounding off problem, there is no mention of the case when the digits dropped are 500$\cdots$

Page 14 gives the dimensions of G as $[M^{-1}L^3T^{-2}]$ while on page 47 the dimensions of G are $[FL^2M^{-2}]$. Both of these forms appear in various textbooks and should be clarified.

This reviewer believes that the monograph is a worthwhile contribution to the resource material for secondary school science teachers and might well be expanded somewhat for reference use by college students in science and engineering.

J. M. C. Hamilton

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

404. *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

On page 89 in Hobson's "Treatise on Plane Trigonometry", 2nd edition, 1897, exercise 1 asks in part to prove that

$$\frac{\sin n\alpha}{\sin \alpha} = 2[\cos(n-1)\alpha + \cos(n-3)\alpha + \cos(n-5)\alpha + \cdots] .$$

Show that this formula holds only for even n and find the correct formula for odd n.

405. *Proposed by C. W. Trigg, Los Angeles City College.*

In how many essentially different ways may 64 congruent cubes, of which 16 are red, 16 are green, 16 are white, and 16 are black, be assembled into a $4\times4\times4$ cube so that no row, column, or pile contains two cubes of the same color?

406. *Proposed by M. N. Gopalan, Mysore City, India.*

If α, β, and γ are the sides of the pedal triangle of a triangle ABC, prove that:

$$\frac{\alpha+\beta+\gamma}{a+b+c} = [\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1] .$$

407. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

a) two adjacent vertices,

b) two opposite vertices of a face,

c) two opposite vertices of the cube.
Which offers the least resistance?

408. *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

Three congruent ellipses are mutually tangent symmetrically. Determine the radius of the circumcircle.

409. *Proposed by D. S. Mitrinovich, University of Belgrade, Yugoslavia.*

Prove

$$(kn)! \equiv 0 \left[\bmod \prod_{r=0}^{n-1} (n+r)\right] \text{ where } n \geqq k .$$

410. *Proposed by Robert W. Kilmoyer, Jr., Lebanon Valley College, Pennsylvania.*

The functions $x(t)$ and $y(t)$ exist along with their derivatives $x'(t)$ and $y'(t)$ respectively. If $x(r^2+s^2) = y(r)y(s)$ where r and s are any independent variables, find $x(t)$ and $y(t)$.

SOLUTIONS

The Explorer Revisited

369. [March and November 1959] *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

An explorer travels on the surface of the earth, assumed to be a perfect sphere, in the manner to be described. First, he travels 100 miles due north. He then travels 100 miles due east. Next he travels 100 miles due south. Finally, he travels 100 miles due west, ending at the point from which he started. Determine all the possible points from which he could have started.

Editor's note: Since the statement of the problem does not exclude the possibility of the explorer retracing a portion of his path, a large family of solutions exists in addition to the one published in November. A number of such solutions have been received since that date.

Solution by Benjamin L. Schwartz, Technical Operations, Inc., Honolulu, Hawaii. Let R denote the radius of the earth, t the length of each segment of the trip. Introduce a spherical coordinate system with origin at the earth's center, and θ and ϕ the longitude and colatitude, respectively. If the explorer starts at $P_0 = (R, \theta_0, \phi_0)$, then by elementary analytic geometry, the succeeding corners of his tour are:

$$P_1 = (R, \theta_0, \phi_0 - \frac{t}{R})$$

$$P_2 = [R, \theta_0 + \frac{t}{R} \sin(\phi_0 - \frac{t}{R}), \phi_0 - \frac{t}{R}]$$

$$P_3 = [R,\ \theta_0 + \frac{t}{R}\sin(\phi_0 - \frac{t}{R}),\ \phi_0]$$

$$P_4 = [R,\ \theta_0 + \frac{t}{R}\sin(\phi_0 - \frac{t}{R}) - \frac{t}{R}\sin\phi_0,\ \phi_0]\ ,$$

and for P_4 to coincide with P_0, we require

$$\theta_4 = \theta_0 + \frac{t}{R}\sin(\phi - \frac{t}{R}) - \frac{t}{R}\sin\phi_0$$

to be coterminal with θ_0 (not necessarily equal to θ_0, as the other solvers have apparently assumed).

We have then

$$\frac{t}{R}[\frac{1}{\sin(\phi_0 - t/R)} - \frac{1}{\sin\phi_0}] = 2k\pi \tag{1}$$

for any integer k. For $k = 0$, we get the published solution.

Other solutions exist, however, for other values of k. Rewrite (1) in the form

$$\frac{\sin\phi_0 - \sin(\phi_0 - t/R]}{\sin\phi_0 \sin(\phi_0 - t/R)} = \frac{2k\pi R}{t} \tag{2}$$

In general, for any fixed integer value of k, this transcendental equation in ϕ_0 has a family of solutions, only a finite number satisfying $0 \leqq \phi_0 \leqq \pi$, which is implied since ϕ is the colatitude. These supplement the previously published partial solution to provide the general solution.

The solutions of (2) are not readily computed in closed form in general. To solve the equation numerically in any particular case, we can use some simple approximations. Since $R >> t$, the right hand side is relatively large when $k \neq 0$, and the numerator of the left hand side is small. Hence, solutions exist only in the neighborhood of $\phi_0 = 0$ or $\phi = \pi$, where the factors of the left side denominator are small. Using first a neighborhood of $\phi_0 = 0$ (the North Pole) we can replace $\sin\phi$ approximately with ϕ, and the equation becomes

$$\frac{t/R}{\phi_0(\phi_0 - t/R)} = \frac{2k\pi R}{t} \tag{3}$$

which can easily be solved as a quadratic in ϕ_0 when numerical values are given to t, R, and k. For example, using $R = 4000$, $t = 100$, $k = 2$, we get

$$\phi_0^2 - 0.0250\,\phi_0 - 0.000049739 = 0 \tag{4}$$

which yields

$$\phi_0 = 0.026852 .$$

This is a circle of latitude 107.41 miles south of the North Pole. The explorer who starts here will proceed north to a point 7.41 miles from the pole; he will then turn eastward and encircle the pole two times, and go on an additional 6.883 miles. A southward journey of 100 miles will return him to his original latitude, exactly 100 miles east of his starting point, and his final westward leg will close the polygon. A similar analysis can be applied to determine the solutions with different numbers of windings around the pole, i. e., different k, as well as those in the neighborhood of the South Pole.

Also solved by Thoger Bang, Copenhagen University Mathematics Institute, Denmark; R. G. Buschman, Oregon State College; A. B. Brown, Queens College, New York; and Henry J. Osner, Modesto Junior College, California.

Disecting a Square

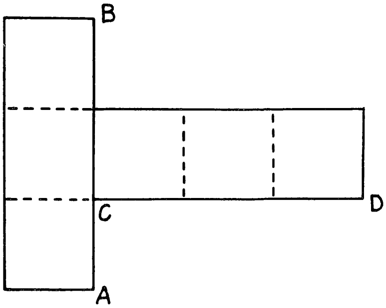
383. [September 1959] *Proposed by Raphael T. Coffman, Richland, Washington.*

Cut any square into not more than six pieces which can be reassembled to form a cube having its surface area equal to the area of the square. Bending of the pieces is permissible.

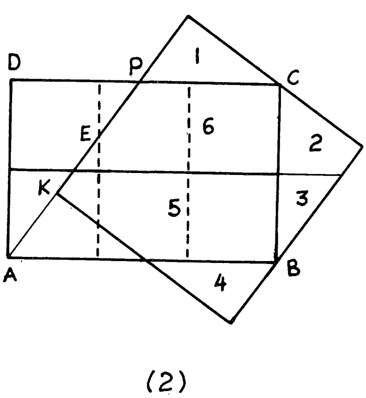
I. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.* Let a be the side of the square. The edge u of the cube is, by $6u^2 = a^2$, $u = a\sqrt{6}/6$. We develop the cube as shown in (1) and assemble the rectangles to form the rectangle $ABCD$ (2). Take $AP = \sqrt{AB \cdot AD} = \sqrt{3u \cdot 2u} = u\sqrt{6}$. Let BE be perpendicular to AB. Then from the similar triangles ABE and APD, having

$$BE : AB = AD : AP \quad \text{and} \quad BE = AB \cdot AD/AP = AP^2/AP = AP ,$$

we can draw the square shown in (2). Comparing (2) and (3) we see the equivalence of $ABCD$ and the square, the side of the latter being evidently a. The number of pieces is 6 and is less than 7.

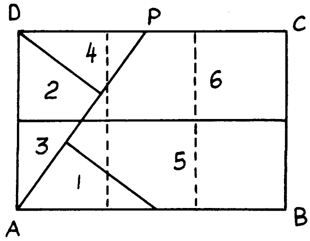


(1)

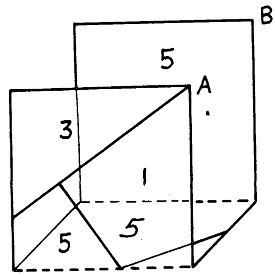


(2)

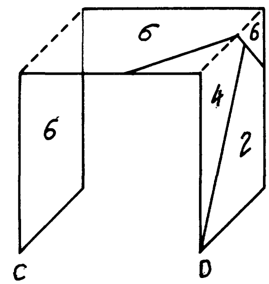
Now, how the cube is obtained is shown by the drawings (3a), (3b) and (4). The solution is therefore completed. If one needs to cut the square into pieces without the use of the rectangle $ABCD$ (2), note the dimensions of (5).



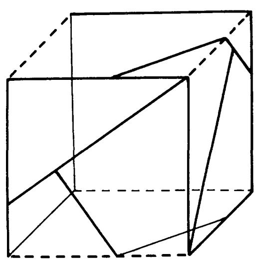
(3)



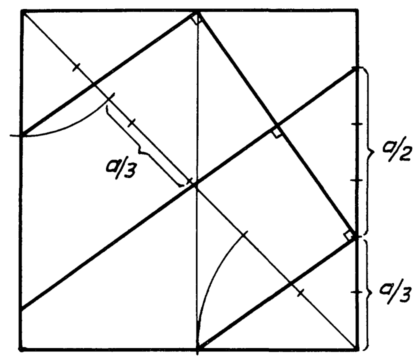
(3a)



(3b)



(4)



(5)

II. *Solution by Maxey Brooks, Sweeny, Texas.* Figure 1 can be folded along the dotted lines to form a cube, 2 units on the edge.

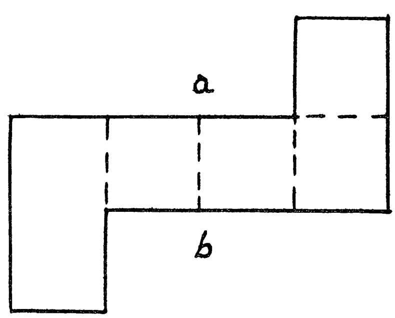


Figure 1

By cutting Figure 1 on line ab, it can be assembled into a 4 by 6 rectangle in Figure 2.

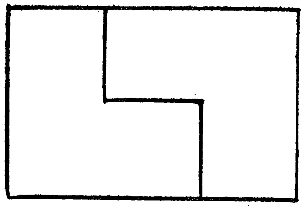


Figure 2

The rectangle can be dissected to form a square $\sqrt{24}$ on the side. The method used is described in Kraitchik, *Mathematical Recreations*, Dover, 1953, pages 193-4.

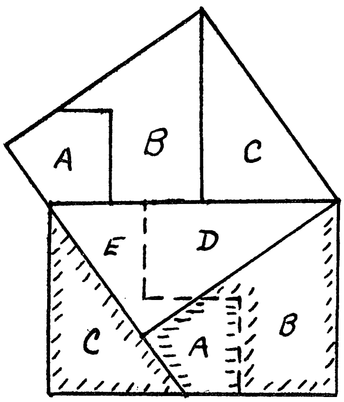


Figure 3

The *five* pieces can be reassembled into the original figure as in Figure 4.

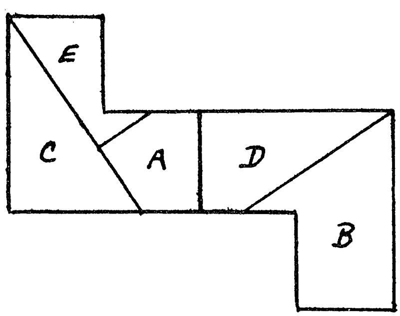


Figure 4

Also solved by the proposer.

An Infinite Group

384. [September 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let (a_{ij}) be a matrix of nth order the sum of the elements of whose rows equals 1. Prove that the totality $[(a_{ij})]$ form a group of infinite order.

Solution by D. A. Breault, Sylvania Electric Products, Inc. We assume that the proposed group operation is multiplication, and that the sum condition means that

$$[1] \qquad \sum_{j=1}^{n} a_{ij} = 1 \quad \text{for} \quad i = 1, 2, \cdots, n$$

The system has
(1) *Closure :* for if $A = [a_{ij}]$, and $B = [b_{ij}]$, we have

$$C_{ij} = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \; ;$$

whence

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{j=1}^{n} b_{kj} \right) = \sum_{k=1}^{n} a_{ik} = 1$$

for each i.
(2) *Associativity :* which can be demonstrated by the use of summations similar to the above.
(3) *Identity :* The usual identity matrix $I = \delta_{ij}$ serves here also.
(4) *Inverses :* given a matrix A, which satisfies [1], it can be shown that A^{-1} satisfies [1] also, *whenever it exists*! Hence the totality of *non-singular* matrices satisfying [1] form a group, but not the unrestricted set.

Also solved by the proposer.

Sums of Three Primes

385. [September 1959] *Proposed by David L. Silverman, Greenbelt, Maryland.*

Find, if they exist:

I. The smallest cube which is the sum of three primes, the sum of any two of which is a square.

II. The largest square which is the sum of three primes, the sum of any two of which is a cube.

Solution by B. L. Meek and J. A. Tyrrell, Kings College, London. We generalize the given problems and try to find positive integral solutions of the equations

$$(1) \qquad p + q + r = n^{\rho}$$

$$(2) \qquad q + r = a^{\sigma}\,, \quad r + p = b^{\sigma}\,, \quad p + q = c^{\sigma}$$

in which p, q and r are primes. We also add the restrictions $\rho > 1$, $\sigma < 1$. (N.B. In case $\sigma = 1$, Goldbach's conjecture ensures any number of solutions: in case $\rho = 1$ the problem is not so trivial, but solutions can certainly be found, e. g. $p = q = r = 2$, $a = b = c = 2$. $n = 6$, $\sigma = 2$).

Reverting to the general case, we first observe that p, q and r cannot all be odd; for, if they were, a, b, and c would all be even by (2) and we could write $a = 2a_1$, $b = 2b_1$, $c = 2c_1$. Addition of the equations (2) would then yield $2(p+q+r) = 2^{\sigma}(a_1^{\sigma} + b_1^{\sigma} + c_1^{\sigma})$, and this is impossible since $p+q+r$ is odd and $\sigma > 1$. We next observe that p, q, and r cannot all be even; for, if so, being prime, they would all be equal to 2 and equation (1) would read $n = 6^{\rho}$ – which cannot be solved in integers with $\rho > 1$.

We can also eliminate the case in which q and r are both odd while $p = 2$; for, in this case, the first of the equations (2) shows that a is even (say $a = 2a_1$) and then elimination with (1) gives $n^{\rho} = 2 + 2^{\sigma}a_1^{\sigma}$. Thus n itself is even (say $n = 2n_1$), $\rho > 1$, $\sigma > 1$.

We are therefore left with only one possible case, viz. that in which $p = q = 2$ and r is odd. In this case, it is easy to show that ρ must be odd and the problem reduces essentially to finding an odd prime p of the form $8k+7$ such that $p+2$ is a perfect square and $p+4$ is a perfect ρ-th power. The least such number is 23.

The answer to problem 385 is therefore as follows:

I. $\rho = 3$, $\sigma = 2$; least solution is $p = q = 2$, $r = 23$.

II. $\rho = 2$, $\sigma = 3$; no solutions.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Maxey Brooke, Sweeney, Texas (Part I); Huseyin Demir, Kandilli, Eregli, Kdz., Turkey, (Part I); Sam Kravitz, East Cleveland, Ohio; Sidney Kravitz, Dover, New Jersey; and the proposer.

A Squared Function

386. [September 1959] *Proposed by Leo Moser, University of Alberta.*

Let $a_i \geqq 0$ and

$$f(x) = a_0 + a_1x + \cdots + a_nx^n .$$

Let

$$g(x) = f^2(x) = b_0 + b_1x + \cdots + b_{2n}x^{2n} .$$

Prove that

$$b_{2r+1} \leqq \frac{1}{2}f^2(1) .$$

I. *Solution by Dale Woods, State Teachers College, Kirksville, Missouri.* Squaring $f(1)$ we have

$$[f(1)]^2 = \sum_{i=0}^{n} a_i^2 + 2a_0\sum_{i=1}^{n} a_i + 2a_1\sum_{i=2}^{n} a_i + \cdots + 2a_{n-1}a_n$$

or

$$[f(1)]^2 = \sum_{i=0}^{n} a_i^2 + 2a_n\sum_{i=1}^{n} a_{n-i} + 2a_{n-1}\sum_{i=2}^{n} a_{n-i} + \cdots + 2a_ia_0$$

But

$$b_{2r+1} = 2\sum_{i=0}^{r} a_i a_{2r+1-i} \text{ if } 2r+1<n \cdot \left\{ b_{2r+1} = 2\sum_{i=0}^{r} a_{n-i} a_{2r+1+i-n} \text{ if } 2r+1>n \right\}$$

Now using the fact that $a_i^2 + a_j^2 \geqq 2a_i a_j$ we have, after proper arrangement, the desired result.

II. *Solution by Charles H. Cunkle, Utah State University.* It is easy to see that $b_{2r+1} \geqq 0$ and

$$\frac{g(x)-g(-x)}{2} = b_1 x + b_3 x^3 + \cdots + b_{2n-1} x^{2n-1}.$$

Then for each r,

$$b_{2r+1} \leqq b_1 + b_3 + \cdots + b_{2n-1} = \frac{g(1)-g(-1)}{2} \leqq \frac{1}{2} g(1) = \frac{1}{2} f^2(1).$$

This is a stronger result than the one proposed.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Dragomir Djoković, Electrical Faculty, University of Belgrade, Yugoslavia; and the proposer.

An Induction Proof

387. [September 1959] *Proposed by D. S. Mitrinovitch, University of Belgrade, Yugoslavia.*

Prove the relation,

$$\left[\frac{\partial^n}{\partial t^n}\left(\frac{1}{1-t} e^{\frac{-xt}{1-t}}\right)\right]_{t=0} = e^x \frac{d^n}{dt^n}(x^n e^{-x})$$

n a natural number, by induction.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey. The equality is evidently true for $n = 0$. Supposing it be true for $n = p$, let us prove it for $n = p+1$. By hypothesis we have

$$\left[\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t} e^{-\frac{xt}{1-t}}\right)\right]_{t=0} = e^x \frac{d^p}{dx^p}(x^p e^{-x}) \tag{1}$$

The right hand side of (1) may be obtained from the Leibniz formula (D stands for d/dx)

$$e^x \frac{d^p}{dx^p}(x^p e^{-x}) = e^x \sum_{k=0}^{p} \binom{p}{k} D^k x^p \cdot D^{p-k} e^{-x} = e^x \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} \frac{p!}{(p-k)!} x^{p-k} e^{-x}$$

$$= e^x \sum_{\lambda=0}^{p} (-1)^{\lambda} \binom{p}{\lambda} \frac{p!}{\lambda!} x^{\lambda} e^{-x}$$

As to the left hand side of (1), we have

$$\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{-\frac{xt}{1-t}}\right) = \frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{x-\frac{x}{1-t}}\right) = e^x\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{-\frac{x}{1-t}}\right)$$

$$= e^x\frac{\partial^p}{\partial t^p}\frac{\partial}{\partial x}e^{-\frac{x}{1-t}} = e^x\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}e^{-\frac{x}{1-t}}.$$

Replacing $u = e^{-\frac{x}{1-t}}$, (1) is reduced to

$$\left[e^x\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}u\right]_{t=0} = e^u\sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p!}{\lambda!}x^\lambda e^{-x}. \tag{2}$$

Now, applying $\frac{\partial}{\partial t}u = x\frac{\partial^2}{\partial x^2}u$ to the left member of (2), we get

$$\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}u = \frac{\partial}{\partial x}\frac{\partial^{p-1}}{\partial t^{p-1}}\left(x\frac{\partial^2}{\partial x^2}\right)u = \frac{\partial}{\partial x}\frac{\partial^{p-2}}{\partial t^{p-2}}\left(x\frac{\partial^2}{\partial x^2}\right)\left(x\frac{\partial^2}{\partial x^2}\right)u = \cdots = \frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p u$$

and

$$\left[\frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p u\right]_{t=0} = \frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p e^{-x} = D(xD^2)^p e^{-x}.$$

The equality reduces therefore to (3)

$$D(xD^2)^p e^{-x} = \sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p!}{\lambda!}x^\lambda e^{-x}. \tag{3}$$

The proof of the statement is completed if we can prove (3) for p replaced by $p+1$. Now,

$$D(xD^2)^{p+1}e^{-x} = D(xD^2)(xD^2)^p e^{-x} = DxD[D(xD^2)^p e^{-x}]$$

$$= DxD\sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p!}{\lambda!}x^\lambda e^{-x} = DxD\sum_{\lambda=0}^{p}a_\lambda x^\lambda e^{-x}$$

$$= (a_1 - a_0) + \sum_{\lambda=1}^{p-1}[(\lambda+1)^2 a_{\lambda+1} - (2\lambda+1)_{a_\lambda + a_{\lambda-1}}]x^\lambda e^{-x} +$$

$$[-(2p+1)a_p + a_{p-1}]x^p e^{-x} + a_p x^{p+1}e^{-x}$$

$$= -(p+1)! + \sum_{\lambda=1}^{p-1}b_\lambda x^\lambda e^{-x} + (-1)^{p-1}(p+1)^2 x^p e^{-x} + (-1)^p x^{p+1}e^{-x} \tag{4}$$

where the first and the last two coefficients are obtained through a_λ, and

$$b_\lambda = (\lambda+1)^2 a_{\lambda+1} - (2\lambda+1)a_\lambda + a_{\lambda-1} \quad (1 \leqq \lambda \leqq p-1)$$

$$= (\lambda+1)^2(-1)^{\lambda+1}\binom{p}{\lambda+1}\frac{p!}{(\lambda+1)!} - (2\lambda+1)(-1)^\lambda\binom{p}{\lambda}\frac{p!}{\lambda!} + (-1)^{\lambda-1}\binom{p}{\lambda-1}\frac{p!}{(\lambda-1)!}$$

$$= (-1)^{\lambda-1}\frac{p!}{\lambda!(p-\lambda+1)!}\frac{p!}{\lambda!}[(p-\lambda)(p-\lambda+1) + (2\lambda+1)(p-\lambda+1) + \lambda^2]$$

$$= (-1)^{\lambda-1}\frac{p!}{\lambda!(p-\lambda+1)!}\frac{p!}{\lambda!}\cdot(p+1)^2 = (-1)^{\lambda-1}\frac{(p+1)!}{\lambda!(p-\lambda+1)!}\frac{(p+1)!}{\lambda!}$$

$$= (-1)^{\lambda-1}\binom{p+1}{\lambda}\frac{(p+1)!}{\lambda!}$$

It is easy to see that the first and the last two coefficients of (4) are b_0, b_p and b_{p+1}, and hence

$$D(xD^2)^{p-1}e^{-x} = \sum_{\lambda=0}^{p+1}(-1)^{\lambda-1}\binom{p+1}{\lambda}\frac{(p+1)!}{\lambda!}x^\lambda e^{-x}.$$

Thus the equality being proved for $n = p+1$, it will be true for all integral values of n and the proof is completed.

Binomial Coefficients

388. [September 1959] *Proposed by M. S. Krick, Albright College, Pennsylvania.*

Prove that

$$\binom{n}{k} = \sum_{s=0}^{t}\binom{t}{s}\binom{n-t}{k-s}, \quad n-t \geqq k \geqq t.$$

Solution by Dragomir Djoković, Electrical Faculty, University of Belgrade, Yugoslavia. [Translated and paraphrased by the editor]: If we equate the proper coefficients of the identity

$$(1+x)^n = (1+x)^t(1+x)^{n-t}$$

we have

$$\sum_{v=0}^{n}\binom{n}{v}x^v = \left[\sum_{v=0}^{t}\binom{t}{v}x^v\right]\left[\sum_{v=0}^{n-t}\binom{n-t}{v}x^v\right].$$

From this identity we deduce the formula

$$\binom{n}{k} = \sum_{j=0}^{k}\binom{y}{s}\binom{n-t}{k-s}$$

which holds for $k = 0, 1, 2, \cdots$. From the fact that $\binom{t}{s} = 0$ for $s = t+1$,

$t+2, \ldots, k$ we arrive at the formula proposed by Krick.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; and the proposer.

Five Wednesdays

389. [September 1959] *Proposed by B. Keshava R. Pai, Belgaum, India.*

There were five Wednesdays in the month of February, 1956, a leap year. During the subsequent century, which years will have five Wednesdays in February?

Solution by Monte Dernham, San Francisco, California. The years sought are of course the leap years in which the first day of February occurs on a Wednesday. Since $365 \equiv 1 \pmod 7$, February the first of any year immediately following an ordinary year will fall on the next succeeding day of the week; immediately following a leap year, it will fall on the second succeeding day of the week. Thus, when the first of February in a leap year occurs on a Wednesday, four years thence, it will fall on the fifth succeeding day, in effect two days earlier in the week, *viz.*, on a Monday. Since 5 (likewise 2) is relatively prime to 7, it requires a cycle of 7 quadrennial leap years to bring February first back to Wednesday; and, since the last year of the present century will be a leap year, 2,000 being divisible by 400, it follows that during the remainder of the 20th and throughout the 21st century every 28th year after 1956 will have the desired property; that is to say, the years

1984, 2012, 2040, 2068, and 2069.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Maxey Brooke, Sweeney, Texas; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Sidney Kravitz, Dover, New Jersey; Wahin Ng, San Francisco, California; C. M. Sidlo, Framingham, Massachusetts; and the proposer.

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 36. Determine the greatest perimeter of all pentagons inscribed in a given circle. [*Submitted by M. S. Klamkin*].

T 37. Given a duodecimal number R_1. Reverse the order of the digits and get a new number R_2. Prove that $|R_1 - R_2|$ is always divisible by e, the eleventh integer. [*Submitted by L. B. Robinson*].

T 38. From a circle of diameter 5 inches, a sector with central angle of $108°$ is cut. The remainder is bent into the curved surface of a right circular cone, the two straight edges touching. Find the slant height of the cone. [*Submitted by Richard K. Guy*].

Solutions

S 38. $2\frac{1}{2}$ inches.

S 37. For such numbers R_1 and R_2 with any base n we have $|R_1 - R_2| \equiv 0 \pmod{n-1}$.

S 36. The perimeter of the regular convex pentagon is $10r\sin\pi/5$, however the perimeter of the regular non-convex pentagon is $10r\sin 2\pi/5$ which is greater. For a $(2n+1)$ sided polygon the maximum perimeter would be $(4n+2)r\sin 2\pi/(2n+1)$, whereas for the $2n$ sided polygon the maximum perimeter would be $4nr$. In this case the polygon has degenerated into the diameter.

FALSIES

A falsie is a problem for which a correct solution is obtained by illegal operations, or an incorrect result is secured by apparently legal processes. For each of the following falsies can you offer an explanation? Send us your favorite falsies for publication.

F 16. In solving the problem 'compute the area bounded by the curves $y = x/(x^2+1)$; $5y = x$ by using double integrals' if we integrate with respect to y first we obtain $\log 5 - (4/5)$. (correct!) But if we integrate with respect to x first, we obtain

$$2\lim_{\epsilon\to 0}\int_{\epsilon}^{2/5}\int_{\frac{1}{2y}-\frac{\sqrt{1-4y^2}}{2y}}^{5y} dy\,dx = 2/5 - \log(4/5)$$

What is wrong in the second answer? [*Submitted by Chih-yi Wang*]

F 17. A student derived L'Hospital's rule in the following manner: Let $\dfrac{F(x)}{G(x)} = H(x)$ where $F(a) = G(a) = 0$ and $G'(a) \neq 0$. Then $F'(x) = G'(x)H(x) + G(x)H'(x)$ or $\dfrac{F'(x)}{G'(x)} = H(x) + \dfrac{G(x)}{G'(x)}H'(x)$. Then $\lim\limits_{x\to a}\dfrac{F'(x)}{G'(x)} = \lim\limits_{x\to a} H(x)$ since $G(a) = 0$ and $G'(a) \neq 0$. [*Submitted by M. S. Klamkin*]

Explanations

E 16. The answer is that the curve C: $y = x/(x^2+1)$ has a maximum at $x = 1$, $y = 1/2$, we have not calculated the portion of the area bounded by C and $y = 2/5$. (Similarly for the other portion between C and $y = 12/5$). Hence if we add the following:

$$2\int_{2/5}^{1/2}\int_{\frac{1}{2y}-\frac{\sqrt{1-4y^2}}{2y}}^{\frac{1}{2y}+\frac{\sqrt{1-4y^2}}{2y}} dy\,dx = -6/5 + 2\log 2$$

to the second answer, we get the correct result.

E 17. Although $G(a) = 0$ and $G'(a) \neq 0$ it does not follow that

$$\lim_{x \to a} \frac{G(x)}{G'(x)} H'(x) = 0 .$$

Actually

$$\frac{G(x)}{G'(x)} H'(x) = \frac{F'(x)}{G'(x)} - \frac{F(x)}{G(x)},$$

so the student's assertion is equivalent to the trivial observation that if

$$\lim_{x \to a}\left[\frac{F'(x)}{G'(x)} - \frac{F(x)}{G(x)}\right] = 0, \quad \text{then} \quad \lim_{x \to a} \frac{F'}{G'} = \lim_{x \to a} \frac{F}{G} .$$

ERRATA

The following diagram was omitted from the answer to Quickie 263 on p. 175, Vol. 33, No. 3, Jan.-Feb., 1960.